STEP/Probability Q5 (12/6/23)

In a simplified game of tennis, a player wins a game by being the first player to win 4 points (ie 15, 30, 40, Game). If the probability that player A wins each point is $\frac{2}{3}$, show that the probability that player A wins the game is $\frac{1808}{2187}$

Solution

Consider the possible cases for the general situation where the probability that player A wins each point is p.

Player B wins no points

$Prob.\ A$ wins game $=\ p^4$

Player B wins 1 point

Possibilities: $BAAAA, ABAAA, AABAA, AAABA$

$Prob.\ A$ wins game $=\ 4p^4q$, where $q = 1 - p$

Player B wins 2 points

Number of ways of selecting 3 As and 2 Bs (followed by an A), where order is important (but the As are indistinguishable, as are the Bs) is $\frac{5!}{3!2!} = \frac{5(4)}{2} = 10$

So $Prob.\ A$ wins game $=\ 10p^3q^2p$

Player B wins 3 points

Number of ways of selecting 3 As and 3 Bs (followed by an A) is $\frac{6!}{3!3!} = \frac{6(5)(4)}{6} = 20$

So $Prob.\ A$ wins game $=\ 20p^3q^3p$

So the probability that player A wins the game is

$p^4 + 4p^4q + 10p^4q^2 + 20p^4q^3$

$=\ p^4(1 + 4q + 10q^2 + 20q^3)$

$=\ p^4(1 + 4 - 4p + 10(1 - p)^2 + 20(1 - p)^3)$

$=\ p^4(35 + p(-4 - 20 - 60) + p^2(10 + 60) - 20p^3)$

$= p^4(35 - 84p + 70p^2 - 20p^3)$

Checks

$p = 0: Prob. = 0$, as expected

$p = 1: Prob. = (35 - 84 + 70 - 20) = 1$, as expected

$p = \frac{1}{2}: Prob. = \left(\frac{1}{2}\right)^4 \left(35 - 42 + \frac{35}{2} - \frac{5}{2}\right) = \frac{8}{16} = \frac{1}{2}$, as expected (by symmetry)

When $p = \frac{2}{3}$, $Prob. = \frac{16}{81}(35 - \frac{168}{3} + \frac{280}{9} - \frac{160}{27})$

$= \frac{16}{81(27)}(945 - 1512 + 840 - 160) = \frac{16(113)}{2187} = \frac{1808}{2187}$, as required.