STEP/Probability Q4 (12/6/23)

An unbiased die has n sides, numbered 1 to n. If the die is thrown twice, find the probability that the score on the 2^{nd} throw is greater than the score on the 1^{st} throw.

Solution

Let S_1 & S_2 be the two scores.

Method 1

$P(S_2 > S_1) = \sum_{i=1}^{n} P(S_1 = i)P(S_2 > i)$

$= \frac{1}{n}(\frac{n-1}{n} + \frac{n-2}{n} + \cdots + \frac{1}{n} + 0)$

$= \frac{1}{n^2} . \frac{1}{2}(n-1)n = \frac{n-1}{2n}$

Method 2

$P(S_2 > S_1) + P(S_2 < S_1) + P(S_2 = S_1) = 1,$

and $P(S_2 > S_1) = P(S_2 < S_1)$, by symmetry.

So $2P(S_2 > S_1) + P(S_2 = S_1) = 1,$

and hence $P(S_2 > S_1) = \frac{1}{2}(1 - P(S_2 = S_1))$

Now, $P(S_2 = S_1) = \sum_{i=1}^{n}(\frac{1}{n})^2 = \frac{n}{n^2} = \frac{1}{n}$,

and so $P(S_2 > S_1) = \frac{1}{2}\left(1 - P(S_2 = S_1)\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{n-1}{2n}$