STEP/Probability Q2 (12/6/23)

The probability that a (biased) coin shows Heads is p, and the probability that it shows Tails is q.

(i) Show that $pq \leq \frac{1}{4}$

(ii) Show that $p^3 + q^3 \geq \frac{1}{4}$

Solution

(i) $pq \le \frac{1}{4} \Leftrightarrow 4p(1-p) \le 1$ (as $p + q = 1$)

$\Leftrightarrow 4p^2 - 4p + 1 \ge 0$

As LHS $= 4(p - \frac{1}{2})^2$, the result is proved.

(ii) $p^3 + q^3 \ge \frac{1}{4} \Leftrightarrow (p+q)^3 - 3p^2q - 3pq^2 \ge \frac{1}{4}$

$\Leftrightarrow 1 - 3pq(p+q) \ge \frac{1}{4}$

$\Leftrightarrow \frac{3}{4} \ge 3pq$

$\Leftrightarrow pq \le \frac{1}{4}$, and this result was established in (i).