STEP/Polynomials Q8 (26/6/23)

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$,

where $n \geq 2$ and the a_i are integers, with $a_0 \neq 0$.

Suppose that there is a rational root $\frac{p}{q}$, where p & q are integers with no common factor greater than 1 and $q > 0$.

By considering $q^{n-1}f(x)$, show that the root will be an integer.
[From STEP 2011, P3, Q2]

Solution

$$\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_2\left(\frac{p}{q}\right)^2 + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

and, multiplying by q^{n-1}:

$$\frac{p^n}{q} + a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1pq^{n-2} + a_1q^{n-1} = 0$$

Then, as all the terms from $a_{n-1}p^{n-1}$ onwards are integers, it follows that $\frac{p^n}{q}$ is also an integer, and hence $q = 1$ (as p & q have no common factor greater than 1), and the root is an integer.