Probability Generating Functions (Discrete RVs)

$G_X(s) = E(s^X) = \sum_{k=0}^{\infty} p_k s^k$

For $B(n,p)$:

$G_X(s) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} s^k = \sum_{k=0}^{n} \binom{n}{k} (ps)^k q^{n-k} = (q+ps)^n$

Find the PGF for $X \sim Po(\lambda)$ $[P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}]$

$$G_X(s) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} s^k = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda}\left(e^{\lambda s}\right) = e^{\lambda(s-1)}$$

$G_X(s) = \sum_{k=0}^{\infty} p_k s^k$

So $G_X'(s) = \sum_{k=0}^{\infty} k p_k s^{k-1}$

and $G_X'(1) = E[X]$

Similarly, $G_X''(1) = E[X(X-1)]$

Find an expression involving the derivatives of $G_X(s)$ for $Var(X)$

$Var(X) = E(X^2) - [E(X)]^2$

$= E[X(X-1)] + E[X] - [E(X)]^2$

$= G_X''(1) + G_X'(1) - [G_X'(1)]^2$

If $X \sim P_o(\lambda)$, prove that $Var(X) = \lambda$, given that $G_X(s) = e^{\lambda(s-1)}$

$G_X(s) = e^{\lambda(s-1)}$

$G'_X(s) = \lambda e^{\lambda(s-1)}$ & $G''_X(s) = \lambda^2 e^{\lambda(s-1)}$

$Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

$= \lambda^2 + \lambda - \lambda^2 = \lambda$

Given that $G_X(s) = \frac{ps}{1-qs}$, what is the distribution of X?

$[G_X(s) = \frac{ps}{1-qs}]$

$G_X(s) = ps(1 + qs + (qs)^2 + \cdots)$

The coefficient of s^k is $p_k = pq^{k-1}$

So X has a Geometric distribution.

If X & Y are independent random variables, then

$G_{X+Y}(s) = G_X(s)G_Y(s)$

Proof

$G_{X+Y}(s) = E(s^{X+Y}) = E(s^X s^Y)$

$= E(s^X)E(s^Y)$ (by independence)

$= G_X(s)G_Y(s)$

PGF of Negative Binomial distribution?

$X = X_1 + \cdots + X_n$, where $X_i \sim Geo(p)$

$G_{X_i}(s) = \frac{ps}{1-qs}$, so $G_X(s) = \left(\frac{ps}{1-qs}\right)^n$

$E(X) = ?$

$G_X(s) = \left(\frac{ps}{1-qs}\right)^n$

So $G_X'(s) = n\left(\frac{ps}{1-qs}\right)^{n-1} . \frac{(1-qs)p-ps(-q)}{(1-qs)^2}$

and $E(X) = G_X'(1) = n\left(\frac{p}{1-q}\right)^{n-1} . \frac{(1-q)p-p(-q)}{(1-q)^2}$

$= n\frac{p}{p^2} = \frac{n}{p}$