STEP/Logarithms Q3 (24/6/23)

(i) Use the graphs of $y = lnx$ and $y = mx$ (for a suitable m) to show that if $e^a = a^e$, then $a = e$.

(ii) Show that, if $a^b = b^a$, where a & b are distinct, then

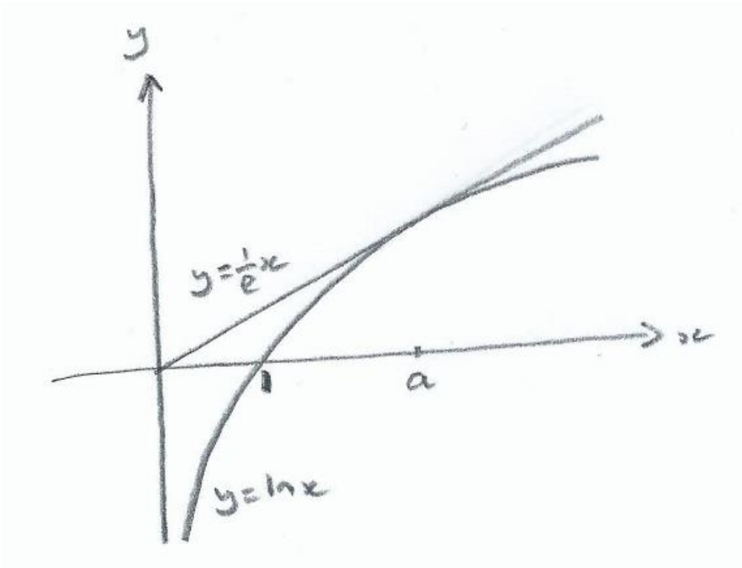
$a < e < b$.

Solution

(i) [Note that if $2^a = a^2$, it doesn't follow that $a = 2$ (as a can equal 4).]

$e^a = a^e \Rightarrow a = elna$ and so $\frac{a}{e} = lna$

Consider the intersection of the graphs $y = lnx$ and $y = \frac{1}{e}x$.



In order for there to be a single solution ($x = a$), $y = \frac{1}{e}x$ must touch $y = lnx$ when $x = a$.

Thus $y = \frac{1}{e}x$ is a tangent to $y = lnx$ at $x = a$,

and so $\frac{d}{dx}(lnx) = \frac{1}{e}$ when $x = a$

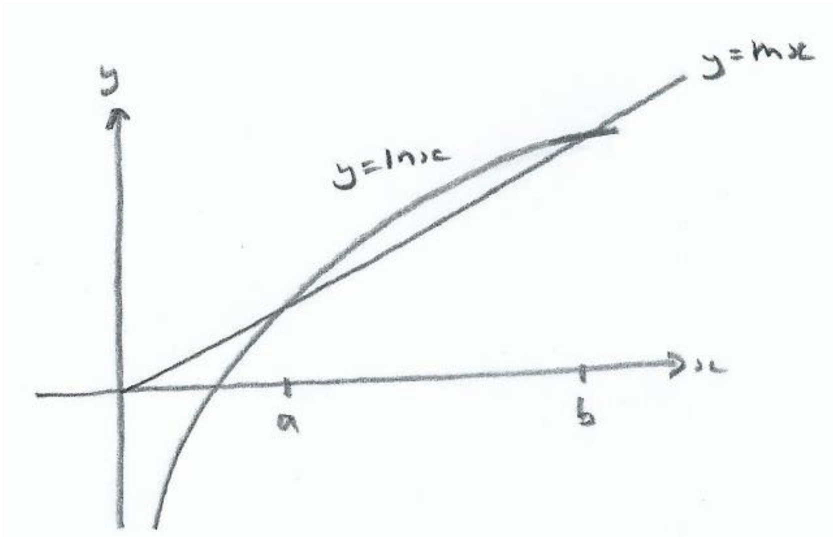
$\Rightarrow \frac{1}{a} = \frac{1}{e}$ and so $a = e$, as required.

(ii) $a^b = b^a \Rightarrow blna = alnb \Rightarrow \frac{lna}{a} = \frac{lnb}{b} = m$, say

Consider the intersection of the graphs $y = lnx$ and $y = mx$.

These occur when $lnx = mx$; ie when $\frac{lnx}{x} = m$,

and so there are points of intersection when $x = a$ & $x = b$.



From (i), the tangent to $y = lnx$ (of the form $y = mx$) occurs when $x = e$, and so $a < e < b$.