STEP/Logarithms Q2 (24/6/23)

Show that $1 - \frac{1}{x} \leq lnx \leq x - 1$, for $x > 0$

Solution

1st part

$\frac{d}{dx}(lnx) = \frac{1}{x}$ and $\frac{d}{dx}\left(1 - \frac{1}{x}\right) = \frac{1}{x^2}$

For $0 < x < 1, \frac{1}{x} < \frac{1}{x^2}$; ie $1 - \frac{1}{x}$ is increasing faster than lnx

For $x > 1, \frac{1}{x} > \frac{1}{x^2}$; ie lnx is increasing faster than $1 - \frac{1}{x}$

[as can be seen from a sketch of the two curves]

When $x = 1$, $lnx = 0$ and $1 - \frac{1}{x} = 0$.

Thus, for $0 < x < 1$, $1 - \frac{1}{x}$ is catching up with lnx, and for $x > 1$, lnx moves away from $1 - \frac{1}{x}$, and hence $lnx \geq 1 - \frac{1}{x}$ for $x > 0$.

2nd part

$lnx \leq x - 1 \Leftrightarrow x \leq e^{x-1} \Leftrightarrow y + 1 \leq e^y$ (1) , where $y = x - 1$

(1) is true for $y = 0$

Now, $\frac{d}{dy}(e^y) = e^y$ and $\frac{d}{dy}(y + 1) = 1$,

so that $\frac{d}{dy}(e^y) \geq \frac{d}{dy}(y + 1)$ for $y \geq 0$

So (1) is true for $y \geq 0$

Now $y + 1 = e^y$ for $y = 0$,

and $\frac{d}{dy}(y + 1) > \frac{d}{dy}(e^y)$ for $y < 0$,

so that $y + 1 < e^y$ for $y < 0$;

ie (1) is true for $y < 0$ as well (though $x > 0 \Rightarrow y > -1$).

Hence $lnx \leq x - 1$ when $x > 0$

So $1 - \frac{1}{x} \leq lnx \leq x - 1$, for $x > 0$