STEP/Integers Q8 (19/2/24)

Let $f(n)$ be the number of factors, other than 1, of the number n.

Show that, if m & n have no common factors,

then $f(mn) = f(m)f(n) + f(m) + f(n)$

Solution

Consider $m = 25$ & $n = 4$. All factors of 100 are either factors of 25, factors of 4, or a product of a factor of 25 and a factor of 4.

Thus the factors of 100 (excluding 1) are formed by combining one number from the set $\{1, 5, 25\}$ and one from the set $\{1, 2, 4\}$, and then discarding the number 1.

[In particular, the factors of 25 are obtained by selecting the 1 from the 2nd set.]

This can be generalised to:

$f(mn) = (f(m) + 1)(f(n) + 1) - 1$

$f(m)f(n) + f(m) + f(n)$, as required.