STEP/Integers Q7 (19/11/23)

Let $h(a, b)$ denote the highest common factor of a & b. Suppose that $b = ka + r$, where k, a & r are positive integers.

Prove that $h(a, b) = h(a, r)$.

Solution

We shall aim to prove that $h(a,b)$ is a divisor of $h(a,r)$, and also that $h(a,r)$ is a divisor of $h(a,b)$.

1st Part

$h(a,b)$ is a divisor of $b = ka + r$,

so that $ka + r = h(a,b)[k\frac{a}{h(a,b)} + \frac{r}{h(a,b)}]$,

where $\frac{a}{h(a,b)} \in \mathbb{Z}^+$, and hence $\frac{r}{h(a,b)} \in \mathbb{Z}^+$ also

(as $k\frac{a}{h(a,b)} + \frac{r}{h(a,b)} \in \mathbb{Z}^+$).

So $h(a,b)$ is a divisor of both a and r; ie a common factor of a and r. And hence $h(a,b)$ is a divisor of $h(a,r)$.

2nd Part

$h(a,r)$ is a divisor of any linear combination of a & r; in particular $ka + r$

So $h(a,r)$ is a common factor of a and $b = ka + r$. And hence $h(a,r)$ is a divisor of $h(a,b)$.

As $h(a,b)$ is a divisor of $h(a,r)$, and $h(a,r)$ is a divisor of $h(a,b)$, it follows that $h(a,b) = h(a,r)$.