STEP/Integers Q5 (21/6/23)

Show that the product of 4 consecutive positive integers is never a perfect square.

Solution

First of all, $(1)(2)(3)(4) = 24$

$(2)(3)(4)(5) = 120$

$(3)(4)(5)(6) = 360$

$(4)(5)(6)(7) = 840$

and we note that $24 = 5^2 - 1$, $120 = 11^2 - 1, 360 = 19^2 - 1$

& $840 = 29^2 - 1$

So, if we can prove that $n(n+1)(n+2)(n+3)+1$ is always a perfect square, then we will have the required result.

The sequence $5, 11, 19, 29$ has 1st differences of $6, 8$ & 10, and 2nd differences of 2, and so is quadratic and of the form

$\frac{1}{2}(2)n^2 + an + b$

Setting n equal to 1 & 2 shows that the sequence is $n^2 + 3n + 1$

So, we want to show that

$n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$

Method 1: Expand both sides

Method 2: Induction

Having shown that the result is true for $n = 1$, we assume that it is true for $n = k$, so that

$k(k+1)(k+2)(k+3) + 1 = (k^2 + 3k + 1)^2$ (A)

We then want to show that, if the result is true for $n = k$, then it will be true for $n = k + 1$;

ie that (B):

$(k+1)(k+2)(k+3)(k+4) + 1 = ([k+1]^2 + 3[k+1] + 1)^2$

By subtracting (A) from (B), this is equivalent to showing (C):

$([k+1]^2 + 3[k+1] + 1)^2 - (k^2 + 3k + 1)^2$

$= (k+1)(k+2)(k+3)(k+4) - k(k+1)(k+2)(k+3)$

LHS of (C) $= \{[k+1]^2 + 3[k+1] + 1 + k^2 + 3k + 1\}$

$\times \{[k+1]^2 + 3[k+1] + 1 - (k^2 + 3k + 1)\}$

$= \{2k^2 + 8k + 6\}\{2k + 4\} = 4(k^2 + 4k + 3)(k+2)$

$= 4(k+1)(k+3)(k+2)$

whilst RHS of (C) $= (k+1)(k+2)(k+3)\{k + 4 - k\}$,

giving the same expression

Thus the result is true for $n = 1$, and if it is true for $n = k$, then it will be true for $n = k + 1$. Hence it must be true for $n = 2, 3$, ..., and therefore all positive integers, by the principle of induction.