STEP/Inequalities Q9 (20/6/23)

Let x, y & z be positive real numbers.

(i) If $x + y \geq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \leq 2$?

(ii) If $x + y \leq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \geq 2$?

Solution

(i) No: if x (say) is very small, then $\frac{1}{x}$ will be very large.

(ii) Note that, when $x = y = 1, \frac{1}{x} + \frac{1}{y} = 2$

Also, if the result is true for $x + y = 2$, then if $x\ or\ y$ is made smaller, so that $x + y < 2, \frac{1}{x} + \frac{1}{y}$ becomes larger, so that the result is still true. So, WLOG (without loss of generality), we need only investigate the case where $x + y = 2$.

[This is an example of "reformulating the problem".]

Experimenting with some numbers, we get the impression that $\frac{1}{x} + \frac{1}{y} \geq 2$. So, aiming for a proof by contradiction, suppose that $\frac{1}{x} + \frac{1}{y} < 2$

Then, $\frac{x+y}{xy} < 2$, so that $2 < 2x(2 - x)$ [as $xy > 0$]

and hence $1 < 2x - x^2$ and $x^2 - 2x + 1 < 0\ \ or\ \ (x - 1)^2 < 0$,

which is impossible.

Thus $\frac{1}{x} + \frac{1}{y} \geq 2$ when $x + y \leq 2$

Alternative approach

To prove that $\frac{1}{x} + \frac{1}{y} \geq 2$ when $x + y = 2$,

we note that WLOG we need only consider solutions of the form $x = 1 + \delta, y = 1 - \delta$ (where $\delta > 0$).

But the reduction from $\frac{1}{1}$ to $\frac{1}{1+\delta}$ will be outweighed by the rise from $\frac{1}{1}$ to $\frac{1}{1-\delta}$ [consider the extreme cases $\frac{1}{1000}\ to\ \frac{1}{1001}$ versus $\frac{1}{4}\ to\ \frac{1}{3}$, which shows that the change of 1 in the denominator has a greater effect when the denominator is smaller, as it is with $1 - \delta$, compared to $1 + \delta$]