

STEP/Inequalities Q8 (20/6/23)

Given that $p, q > 0$ and that $p \neq q$, show that

$$p^{2n}q + q^{2n}p > (pq)^nq + (qp)^np$$

Solution

$$\begin{aligned} & p^{2n}q + q^{2n}p - (pq)^nq - (qp)^np \\ &= p^nq(p^n - q^n) + q^n p(q^n - p^n) \\ &= (p^n - q^n)(p^nq - q^n p) \\ &= (p^n - q^n)pq(p^{n-1} - q^{n-1}) \\ &= (p - q)(p^{n-1} + qp^{n-2} + \dots + q^{n-1})pq \\ &\cdot (p - q)(p^{n-2} + qp^{n-3} + \dots + q^{n-2}) \\ &> 0, \text{ as } (p - q)^2 > 0 \text{ (as } p \neq q) \end{aligned}$$