STEP/Inequalities: Exercises - Overview (20/6/23)

Q1

Are the following true or false?

(i) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

(ii) $a < b \Rightarrow a^2 < b^2$

(iii) $a < b \; \& \; c < d \Rightarrow a + c < b + d$

(iv) $a < b \; \& \; c < d \Rightarrow a - c < b - d$

Q2

Assuming that $sin^2\theta + cos^2\theta = 1$, but without using any compound angle results, show that $sin\theta cos\theta \leq \frac{1}{2}$

Q3

Is $\frac{6}{7} < \frac{2}{\sqrt{5}}$? [without using a calculator]

Q4

Which is larger: $\frac{\sqrt{7}}{2}$ or $\frac{1+\sqrt{6}}{3}$ (without using a calculator)?

Q5

Show that $e^3 > 4e^{\frac{3}{2}}$

Q6

Prove that, for $a, b, c > 0$, $\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a < b$

Q7

Prove or provide a counter-example for the conjecture

$x > a \ \& \ y > b \Rightarrow xy > ab \quad (a, b \ real)$ in each of the following cases:

(i) $a > 0, b > 0$ (ii) $a < 0, b < 0$ (iii) $a > 0, b < 0$

Q8

Given that $p, q > 0$ and that $p \neq q$, show that

$p^{2n}q + q^{2n}p > (pq)^n q + (qp)^n p$

Q9

Let $x, y \ \& \ z$ be positive real numbers.

(i) If $x + y \geq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \leq 2$?

(ii) If $x + y \leq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \geq 2$?

Q10

Show that if $X > 1 \ \& \ Y > 1$, then $X + Y < XY + 1$

Q11

Given that $p, q > 0$ and that $p \neq q$, show that

$$p^{2n}q + q^{2n}p > (pq)^n q + (qp)^n p$$