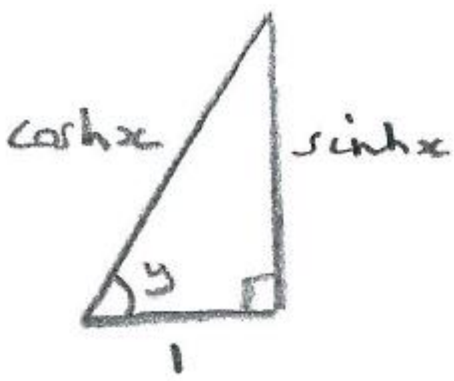
STEP/Hyperbolic Functions Q4 (16/6/23)

Given that $sinhx = tany$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$, show that

(a) $tanhx = siny$ (b) $x = \ln\,(tany + secy)$

Solution

(a) As $sinhx = tany$, we can construct a right-angled triangle (see diagram below), where the hypotenuse is $coshx$, as $sinh^2x + 1 = cosh^2x$.



Then $siny = \frac{sinhx}{coshx} = tanhx$, as required.

Alternatively: $tanhx = \frac{sinhx}{coshx} = \frac{tany}{\sqrt{1+sinh^2x}}$

(from $sinh^2x + 1 = cosh^2x$, noting that $coshx$ is always positive, so that we take the positive square root)

$= \frac{tany}{\sqrt{1+tan^2y}} = \frac{tany}{\sqrt{sec^2y}} = \frac{tany}{secy}$

(as $cosy > 0$ when $-\frac{\pi}{2} < y < \frac{\pi}{2}$, and hence $secy > 0$ also)

$= tanycosy = siny$

(b) From the right-angled triangle,

$tany + secy = sinhx + coshx$

$= \frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^x + e^{-x}) = e^x$,

so that $\ln(tany + secy) = x$, as required.

Alternatively: $sinhx = tany \Rightarrow \frac{1}{2}(e^x - e^{-x}) = tany$

$\Rightarrow e^{2x} - 1 = 2tanye^x$

$\Rightarrow e^{2x} - 2tanye^x - 1 = 0$

$\Rightarrow e^x = \frac{2tany \pm \sqrt{4tan^2y+4}}{2} = tany \pm secy$

$tany - secy = \frac{siny-1}{cosy} < 0$ when $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Hence, as $e^x > 0$, it follows that $e^x = tany + secy$,

and hence $x = \ln(tany + secy)$