STEP/Hyperbolic Functions Q2 (16/6/23)

Given that $artanhx = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right)$ and $arcothx = \frac{1}{2}ln\left(\frac{1+x}{x-1}\right)$,

and also that $\frac{d}{dx}(artanhx) = \frac{d}{dx}(arcothx) = \frac{1}{1-x^2}$,

what is wrong with the following reasoning?

$\int \frac{1}{1-x^2}\ dx = artanhx + C = arcothx + C_1$,

so that $artanhx - arcothx = C_2$

But $artanhx - arcothx = \frac{1}{2}ln\left(\frac{\left(\frac{1+x}{1-x}\right)}{\left(\frac{1+x}{x-1}\right)}\right) = \frac{1}{2}ln\left(\frac{x-1}{1-x}\right) = \frac{1}{2}\ln(-1)$,

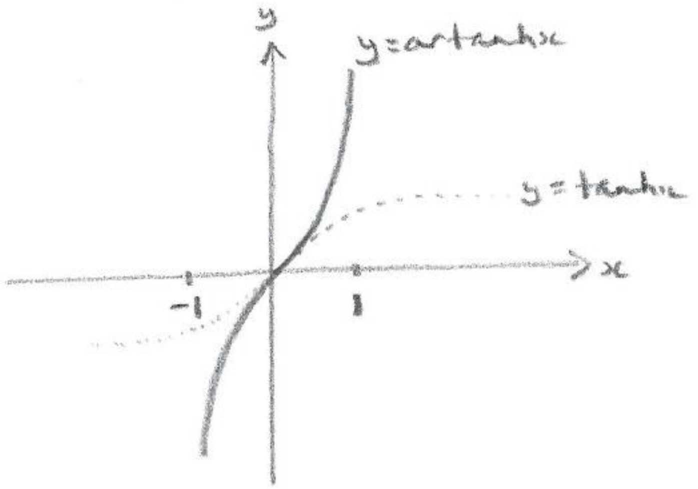
which isn't defined!

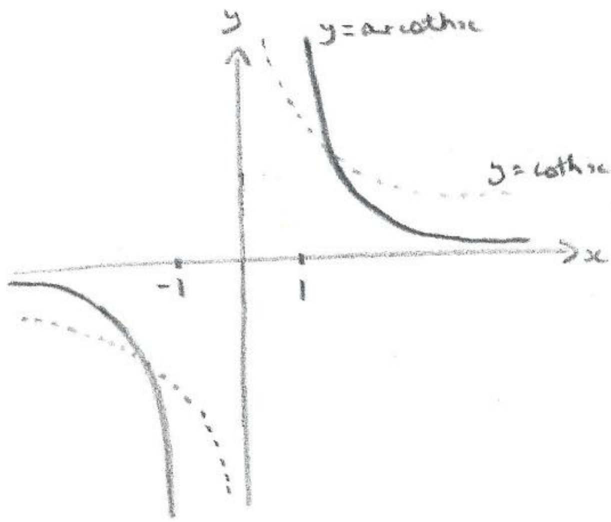
Solution

The problem is that the domains of $y = artanhx$ and

$y = arcothx$ don't overlap (see graphs below). We ought to say that $artanhx = \frac{1}{2} ln \left(\frac{1+x}{1-x}\right)$ for $|x| < 1$ and $arcothx = \frac{1}{2} ln \left(\frac{1+x}{x-1}\right)$ for $|x| > 1$. So it doesn't make sense to determine

$artanhx - arcothx$





Note that, with $|x| < 1$, $\frac{d}{dx}(artanhx) = \frac{1}{1-x^2} > 0$ for all x; whilst

with $|x| > 1$, $\frac{d}{dx}(arcothx) = \frac{1}{1-x^2} < 0$ for all x