STEP/Geometry Q1 (16/6/23)

Prove that, if a straight line bisects the area of a square, then it passes through the centre of the square.

Solution

Let the square have corners at (0,0), (1,0), (1,1)& (0,1).

The line must cross either the two vertical sides or the two horizontal sides, otherwise it will lie to one side of the diagonal, and will not bisect the area of the square.

Without loss of generality, suppose that the line crosses the two vertical sides, and that it does this at $(0, a)$ and $(1, b)$.

The area below the line is then $\frac{1}{2}(a + b)(1)$, and as this must equal half of the area of square, $\frac{1}{2}(a + b) = \frac{1}{2}(1)$, so that

$b = 1 - a$, and the eq'n of the line is $\frac{y-a}{x-0} = \frac{a-(1-a)}{0-1}$

or $y = a - x(2a - 1)$

Then, when $x = \frac{1}{2}, y = \frac{1}{2}$, so that the line passes through the centre of the square, as required.