STEP/General Q3 (13/6/23)

(i) Find an expansion for $(a+b+c)^3$, and give a justification for the coefficients.

(ii) Extend this to $(a+b+c)^4$

Solution

(i) By an ordinary expansion:

$(a+b+c)^3 = ([a+b]+c)^3$

$= (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$

$= (a^3 + 3a^2b + 3ab^2 + b^3) + (3a^2c + 3b^2c + 6abc)$

$+(3ac^2 + 3bc^2) + c^3$

$= (a^3 + b^3 + c^3) + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$

$+6abc$

Alternatively, this could have been deduced by noting that the terms fall into one of the 3 groups above.

Then there is only 1 way of creating an a^3 term from

$(a+b+c)(a+b+c)(a+b+c)$; namely by choosing the a from each of the 3 brackets.

There are 3 ways of creating an a^2b term: 3[number of ways of choosing the b]$\times$ 1[number of ways of choosing two as from the remaining 2 brackets].

Finally, there are 6 ways of creating an abc term: 3[number of ways of choosing the a]$\times$ 2[number of ways of choosing the b from the remaining 2 brackets]$\times$ 1[number of ways of choosing the c from the remaining bracket].

The final expression then follows by symmetry.

(ii) $(a+b+c)^4 = (a^4 + b^4 + c^4)$

$+4(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b)$

$+6(a^2b^2 + a^2c^2 + b^2c^2) + 12(a^2bc + b^2ac + c^2ab)$

For the a^2b^2 term etc, there are $\binom{4}{2} = 6$ ways of choosing the brackets from $(a+b+c)(a+b+c)(a+b+c)(a+b+c)$

to give a^2, and then just 1 way of obtaining the b^2 term.

For the a^2bc term etc, there are $\binom{4}{2} = 6$ ways of choosing the brackets for the a^2 again, multiplied by the 2 ways of choosing brackets for the b and c.

For further investigation: the 'trinomial' expansion of $(a+b+c)^n$ can be shown to be $\sum_{\substack{i,j,k \\ (i+j+k=n)}} \binom{n}{i,j,k} a^i b^j c^k$,

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

(with a further extension to the 'multinomial' expansion of

$(a_1 + a_2 + \cdots + a_m)^n$)