STEP/Forces, Q5 (11/6/23)

A rollercaster ride is modelled by a particle on a smooth wire. If a point on the wire has coordinates (x, y), show that

$\dot{x}\ddot{x} + \dot{y}(\ddot{y} + g) = 0$

(a) by an energy method, and (b)(as an alternative method)

by applying Newton's 2nd Law

Solution

(a) By Conservation of Energy,

$KE + PE = C$, where C is a constant;

ie $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy = C$, where m is the mass of the particle.

Differentiating wrt t, $m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + mg\dot{y} = 0$,

so that $\dot{x}\ddot{x} + \dot{y}(\ddot{y} + g) = 0$, as required.

(b) As the wire is smooth, the only force on the particle affecting its motion is the component of its weight along the wire.

By N2L, $-mgsin\theta = m(\ddot{x}cos\theta + \ddot{y}sin\theta)$,   [see Note below]

where the gradient of the wire is $\frac{dy}{dx} = tan\theta$ at the point (x, y), and $\ddot{x}$ & $\ddot{y}$ are the x & y components of the acceleration of the particle.

Then $-gtan\theta = \ddot{x} + \ddot{y}tan\theta$,

and hence, since $tan\theta = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}$,

$-g\dot{y} = \dot{x}\ddot{x} + \ddot{y}\dot{y}$, or $\dot{x}\ddot{x} + \dot{y}(\ddot{y} + g) = 0$, as required.

[Note: If instead the force along the wire is resolved in the x & y directions:

$(-mgsin\theta)cos\theta = m\ddot{x}$ and $(-mgsin\theta)sin\theta = m\ddot{y}$

Then $m(\ddot{x}cos\theta + \ddot{y}sin\theta) = (-mgsin\theta)(cos^2\theta + sin^2\theta)$

$= -mgsin\theta$, as before.]