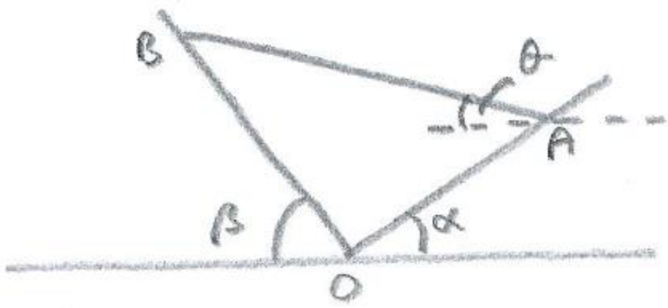
STEP/Forces, Q1 (11/6/23)

A uniform rod AB lies in equilibrium between two smooth planes inclined at angles α and β to the horizontal, as shown in the diagram, where $\beta > \alpha$, such that the vertical plane containing AB is perpendicular to the line of intersection of the two planes.

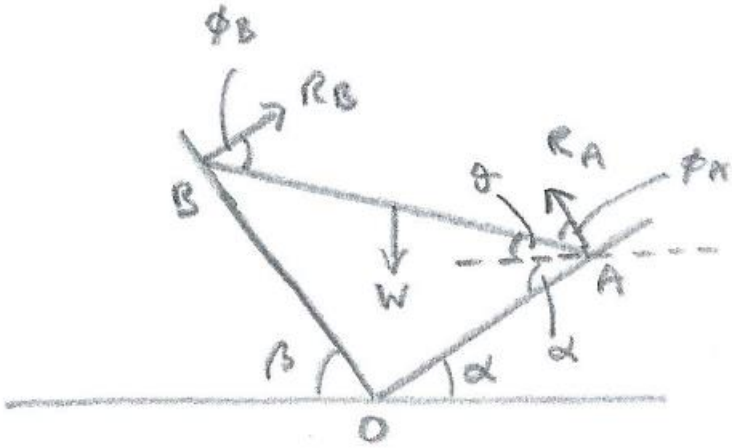
(i) Show that the ratio of the reactions at A and B is $sin\beta : sin\alpha$

(ii) If AB makes an angle θ to the horizontal, show that

$tan\theta = \frac{\sin(\beta - \alpha)}{2sin\alpha sin\beta}$



Solution



(i) Taking moments about the centre of mass of AB, which is assumed to be of length $2d$,

rotational equilibrium $\Rightarrow R_A sin\phi_A d = R_B sin\phi_B d$,

And $\phi_A + \theta + \alpha = 90°$ (the angle that R_A makes with OA)

& $\phi_B + (180° - [180° - \alpha - \beta] - [\alpha + \theta]) = 90°$

(the angle that R_B makes with OB),

and so $\phi_B - \theta + \beta = 90°$

Then $\frac{R_A}{R_B} = \frac{sin\phi_B}{sin\phi_A} = \frac{cos(90°-\phi_B)}{cos(90°-\phi_A)} = \frac{cos(\beta-\theta)}{cos(\alpha+\theta)}$ (1)

[Instead of resolving in 2 perpendicular directions, we can (if necessary) obtain 2 equations from Lami's theorem:]

As AB is in equilbrium, the triangle of forces can be applied (see diagram, where W is the weight of AB). Then, by Lami's theorem:

$\frac{R_A}{sin\gamma_A} = \frac{R_B}{sin\gamma_B}$ (2)

[A similar equation involving W could also be obtained, but this introduces a further unknown (ie W) into the equations.]

Drawing a vertical line through B gives

$\gamma_A + 90° + (90° - \beta) = 180°$, so that $\gamma_A = \beta$

Drawing a vertical line through A gives

$\gamma_B + 90° + (90° - \alpha) = 180°$, so that $\gamma_B = \alpha$

Then, from (2), $\frac{R_A}{R_B} = \frac{sin\gamma_A}{sin\gamma_B} = \frac{sin\beta}{sin\alpha}$, as required.

(ii) From (1), $\frac{cos(\beta-\theta)}{cos(\alpha+\theta)} = \frac{sin\beta}{sin\alpha}$, so that

$sin\alpha(cos\beta cos\theta + sin\beta sin\theta) = sin\beta(cos\alpha cos\theta - sin\alpha sin\theta)$

and hence (dividing by $cos\theta$),

$sin\alpha cos\beta + sin\alpha sin\beta tan\theta = sin\beta cos\alpha - sin\beta sin\alpha tan\theta$, so that $tan\theta(2sin\alpha sin\beta) = sin\beta cos\alpha - sin\alpha cos\beta$,

and $tan\theta = \frac{\sin{(\beta-\alpha)}}{2sin\alpha sin\beta}$, as required.