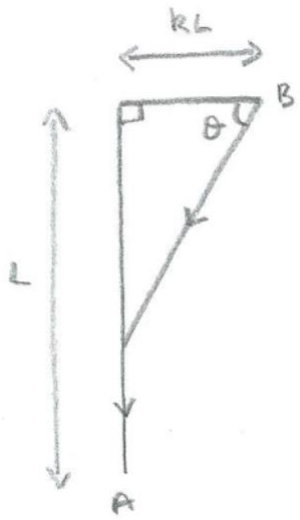
STEP/Differentiation Q7 (15/6/23)

A dog is being taken for a walk on a path round the edge of a ploughed field. The owner starts at A (see diagram), and walks it a distance L along one side of the field, and then (after turning a right angle) a distance kL along the next side. At B, the dog is let off the lead, but decides to run back to A, along the route indicated by arrows on the diagram (ie a stretch of ploughed field, followed by a stretch of path). If the dog's speed is reduced by λ% when running on the ploughed field, compared with the path, find an expression for the angle θ that minimises the time taken for it to return to A.



Solution

The time taken by the dog over each stretch is inversely proportional to its speed, and so the total time taken by the dog is proportional to

$T = (L - kLtan\theta) + \frac{1}{1-\frac{\lambda}{100}} . \frac{kL}{cos\theta}$ [3 marks]

Writing $f = \frac{1}{1-\frac{\lambda}{100}}$, a stationary point for the time occurs when

$\frac{dT}{d\theta} = 0$, so that $-kLsec^2\theta + fkLsec\theta tan\theta = 0$ [2 marks]

and, as $sec\theta \neq 0$, $-sec\theta + ftan\theta = 0$

$\Rightarrow fsin\theta = 1,$ [1 mark]

so that $sin\theta = 1 - \frac{\lambda}{100}$,

and $\theta = \arcsin(1 - \frac{\lambda}{100})$, as $0 < \theta < 90°$ [2 marks]

[Note that this doesn't depend on k.]

[Check: If $\lambda = 100$, we would expect the dog to not cut a corner at all; ie $\theta = 0°$]