STEP: Differential Equations (6 pages; 21/4/23)

See also Exercises - Differential Equations.

(1) **Exercise**: Solve $\frac{dy}{dx} = x + y$ by:

(a) making the substitution $z = x + y$

(b) finding an integrating factor

Solution

(a) $\frac{dy}{dx} = x + y \Rightarrow \frac{d}{dx}(z - x) = z$

$\Rightarrow \frac{dz}{dx} - 1 = z$

$\Rightarrow \frac{dz}{dx} = z + 1$

$\Rightarrow \int \frac{1}{z+1}\, dz = \int dx$

$\Rightarrow \ln|z + 1| = x - lnC$

$\Rightarrow C(z + 1) = e^x$

$\Rightarrow y = z - x = Ae^x - 1 - x$

(b) $\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - y = x$

I.F. $= \exp\{\int -1\, dx\} = e^{-x}$

Then $e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{-x}$

$\Rightarrow \frac{d}{dx}(ye^{-x}) = xe^{-x}$

$\Rightarrow ye^{-x} = \int xe^{-x}\, dx = x(-e^{-x}) - \int -e^{-x}\, dx = -xe^{-x} - e^{-x} + C$

$\Rightarrow y = Ce^x - 1 - x$

(2) To convert $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$

to $\frac{d^2y}{du^2} + c\frac{dy}{du} + dy = 0$ (*)

Exercise: Which of the following substitutions works:

$u = e^x$ or $x = e^u$?

Solution

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Now, $u = e^x \Rightarrow \frac{du}{dx} = u$,

and $x = e^u \Rightarrow \frac{du}{dx} = \frac{1}{\left(\frac{dx}{du}\right)} = \frac{1}{x}$

In the latter case, $\frac{dy}{dx} = \frac{dy}{du}(\frac{1}{x})$, and $x\frac{dy}{dx} = \frac{dy}{du}$

Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\left(\frac{1}{x}\right)\right) = \left(\frac{d^2y}{du^2}\,.\frac{du}{dx}\right)\left(\frac{1}{x}\right) + \frac{dy}{du}(-\frac{1}{x^2})$

$= \frac{1}{x^2}(\frac{d^2y}{du^2} - \frac{dy}{du})$

So $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$ becomes

$\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + a\frac{dy}{du} + by = 0$

ie $\frac{d^2y}{du^2} + (a-1)\frac{dy}{du} + by = 0$

(3) **Exercise**:

(i) Show that $\frac{dy}{dx} = f(\frac{y}{x})$ can potentially be solved by making a substitution.

(ii) Solve $\frac{dy}{dx} = \frac{x^3+4y^3}{3xy^2}$, $x > 0$

Solution

(i) Let $z = \frac{y}{x}$, so that $y = xz$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$

So $\frac{dy}{dx} = f(\frac{y}{x})$ becomes $z + x\frac{dz}{dx} = f(z)$

and $\int \frac{1}{f(z)-z} dz = \int \frac{1}{x}\, dx$

(ii) Let $z = \frac{y}{x}$, so that $\frac{dy}{dx} = z + x\frac{dz}{dx}$, as in (i).

Then $z + x\frac{dz}{dx} = \frac{1}{3z^2} + \frac{4z}{3}$

and $x\frac{dz}{dx} = \frac{1}{3z^2} + \frac{z}{3}$

so that $3\int \frac{1}{\frac{1}{z^2}+z}\, dz = \int \frac{1}{x} dx$

and $lnx = \int \frac{3z^2}{1+z^3} dz = \ln(1 + z^3) + lnC$

$\Rightarrow x = C(1 + z^3) \quad [C > 0]$

$\Rightarrow (\frac{y}{x})^3 = Ax - 1 \quad [A = \frac{1}{C}]$

$\Rightarrow y^3 = (Ax - 1)x^3$

[Further example: $(x-y)(x+y)\frac{dy}{dx} = xy$]

(4) (i) Extend this approach to DEs related to $\frac{dy}{dx} = x + y$

(ii) Solve $\frac{dy}{dx} = (x+y)(x+y-2)$

Solution

(i) If $z = x + y$, then $\frac{dz}{dx} = 1 + \frac{dy}{dx}$,

so that $\frac{dy}{dx} = f(x+y)$ becomes $\frac{dz}{dx} - 1 = f(z)$

and $\frac{dz}{dx} = f(z) + 1$ is potentially solvable by separation of variables.

[Similarly for $\frac{dy}{dx} = f(ax+by)$; eg $\frac{dy}{dx} = \frac{-(1+2y+4x)}{1+y+2x}$]

(ii) As the RHS is a function of $x + y$, let $z = x + y$

Then $\frac{dz}{dx} - 1 = z(z-2)$,

so that $\frac{dz}{dx} = z^2 - 2z + 1 = (z-1)^2$

$\Rightarrow \int \frac{1}{(z-1)^2} dz = \int dx$

$\Rightarrow -\frac{1}{(z-1)} = x + C$

$\Rightarrow x + y - 1 = -\frac{1}{(x+C)}$

$\Rightarrow y = 1 - x - \frac{1}{(x+C)}$

(5) Substitutions

$\frac{dy}{dx} = f(x + y)$

Let $z = x + y$, so that $\frac{dy}{dx} = \frac{dz}{dx} - 1$ can be obtained

$\Rightarrow \frac{dz}{dx} = f(z) + 1$ (& separation of variables can be applied)

$\frac{dy}{dx} = f\left(\frac{x}{y}\right)$

Let $z = \frac{x}{y}$, so that $\frac{dy}{dx} = z + x\frac{dz}{dx}$ can be obtained

$\Rightarrow x\frac{dz}{dx} = f(z) - z$ (& separation of variables can be applied)

(6) Exercise: Solve $\frac{dy}{dx} + xy = xy^2$ by means of the substitution

$z = \frac{1}{y}$

Solution

$y = \frac{1}{z} \Rightarrow \frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$

so that $\frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$

and $\frac{dy}{dx} + xy = xy^2$ becomes $-\frac{dz}{dx} + xz = x$

and then an I.F. can be found.

Notes

(i) So $z = \frac{1}{y}$ is potentially useful for a DE of the form $\frac{1}{y^2}\frac{dy}{dx} + \cdots$

(ii) In general, $y^n \frac{dy}{dx}$ suggests $z = y^{n+1}$

(and $y^{-n} \frac{dy}{dx}$ suggests $z = y^{-n+1}$)

In fact, $\frac{dy}{dx} + P(x)y = Q(x)y^n$ can be transformed to

$\frac{dz}{dx} - (n-1)P(x).z = -(n-1)Q(x)$ by $z = y^{-n+1}$