STEP/Differential Equations Q2 (15/6/23)

To convert $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$

to $\frac{d^2y}{du^2} + c\frac{dy}{du} + dy = 0$ (*)

Which of the following substitutions works: $u = e^x$ or $x = e^u$?

Solution

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Now, $u = e^x \Rightarrow \frac{du}{dx} = u$,

and $x = e^u \Rightarrow \frac{du}{dx} = \frac{1}{\left(\frac{dx}{du}\right)} = \frac{1}{x}$

In the latter case, $\frac{dy}{dx} = \frac{dy}{du}(\frac{1}{x})$, and $x\frac{dy}{dx} = \frac{dy}{du}$

Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\left(\frac{1}{x}\right)\right) = \left(\frac{d^2y}{du^2}\cdot\frac{du}{dx}\right)\left(\frac{1}{x}\right) + \frac{dy}{du}(-\frac{1}{x^2})$

$= \frac{1}{x^2}(\frac{d^2y}{du^2} - \frac{dy}{du})$

So $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$ becomes

$\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + a\frac{dy}{du} + by = 0$

ie $\frac{d^2y}{du^2} + (a-1)\frac{dy}{du} + by = 0$