STEP/Curve Sketching Q4 (14/6/23)

Sketch $x^n \pm y^n = 1$ for large n

Solution

Consider odd and even n separately.

(i) $x^n - y^n = 1$; odd n

For large positive x & y, $y^n = x^n - 1$, so that $y \approx x$; ie $y = x$ is an asymptote. Also for large negative x & y.

Negative x with positive y is not possible.

Positive x with negative y is only possible if both $|x|$ & $|y|$ are < 1.

Curve passes through $(0, -1)$ and $(1, 0)$.

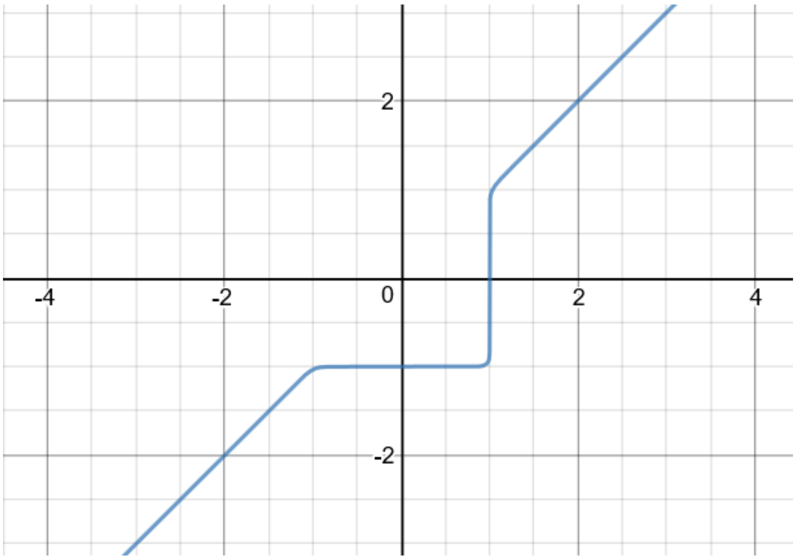
When $|x| < 1, x^n$ can be made as small as we like, by making n large enough. Then $x^n - y^n = 1 \Rightarrow y \to -1$ as $n \to \infty$.

Similarly, for $|y| < 1, x \to 1$ as $n \to \infty$.

For $x > 1, x^n$ can be made as large as we like, by making n large enough. Then $x^n - y^n = 1 \Rightarrow y \to x$ as $n \to \infty$.

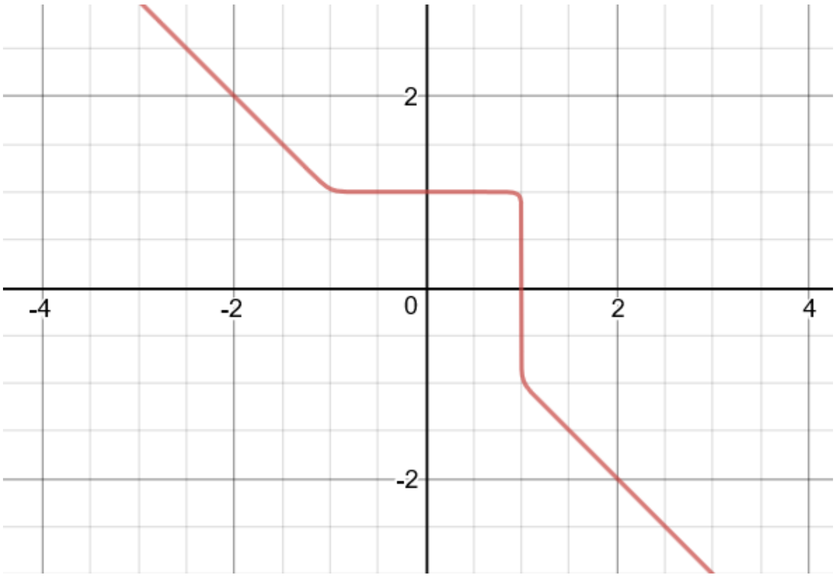
For $x < -1, x^n$ can be made as large and negative as we like (as n is odd), by making n large enough. Then $x^n - y^n = 1 \Rightarrow y \to x$ as $n \to \infty$.

Finally, $x = -1 \Rightarrow y^n = -2 \Rightarrow y \to -1$ as $n \to \infty$.



(ii) $x^n + y^n = 1$; odd n

similar reasoning can be applied



(iii) $x^n + y^n = 1$; even n

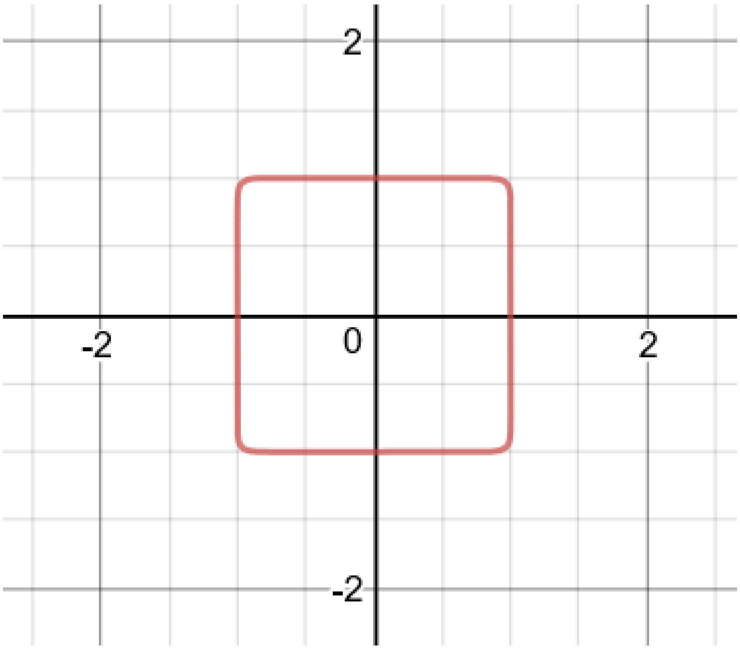
Curve has symmetry about the y-axis (replacing x with $-x$) and about the x-axis (replacing y with $-y$).

As n is even, $x^n + y^n = 1$ has no solution for $|x| > 1\ or\ |y| > 1$.

Curve passes through $(0,1), (0,-1), (1,0)$ & $(-1,0)$.

When $|x| < 1, x^n$ can be made as small as we like, by making n large enough. Then $x^n + y^n = 1 \Rightarrow y \to \pm 1$ as $n \to \infty$.

Similarly, for $|y| < 1, x \to \pm 1$ as $n \to \infty$.



[This is a straightened out version of the circle $x^2 + y^2 = 1$]

(iv) $x^n - y^n = 1$; even n

Once again, the curve has symmetry about the x and y-axes.

For large $|x|$ & $|y|$, $y^n = x^n - 1 \Rightarrow y \approx \pm x$; ie $y = \pm x$ are asymptotes.

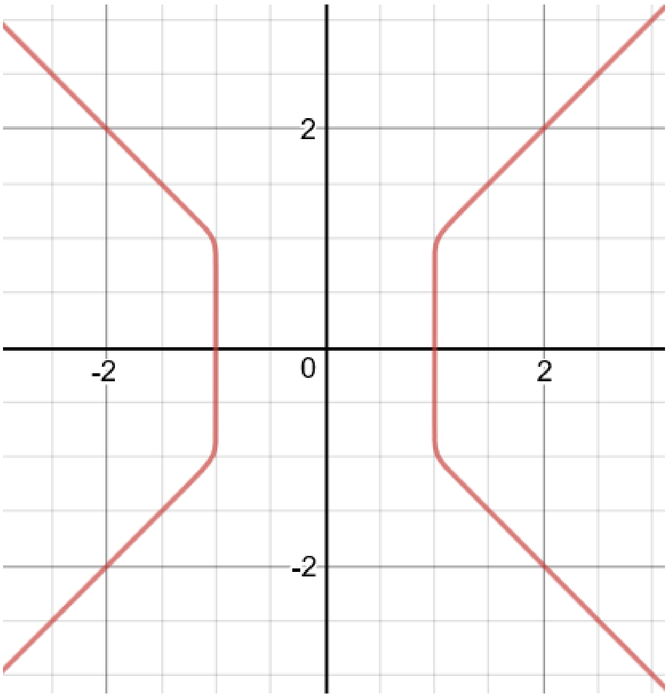
Curve passes through $(1, 0)$ and $(-1, 0)$, but doesn't cross the

y-axis.

When $|x| < 1, x^n$ can be made as small as we like, by making n large enough. Then $x^n - y^n = 1 \Rightarrow$ no solution.

Similarly, for $|y| < 1$, $x \to 1$ as $n \to \infty$.

For $x > 1$, x^n can be made as large as we like, by making n large enough. Then $x^n - y^n = 1 \Rightarrow y \to x$ as $n \to \infty$.



[This is a straightened out version of the rectangular hyperbola $x^2 - y^2 = 1$]