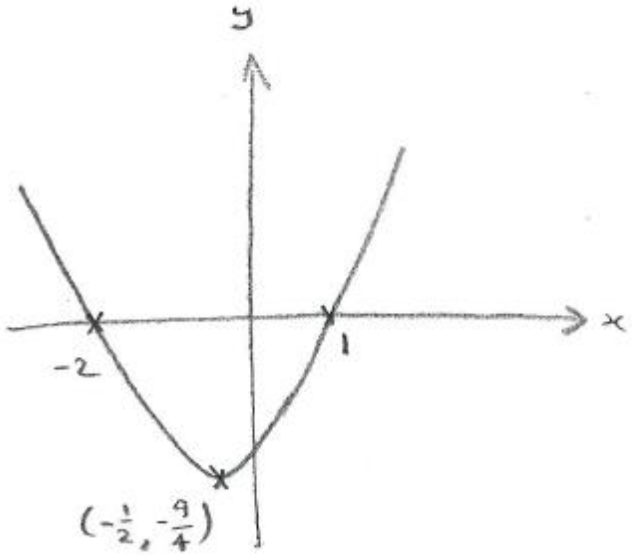
Find the turning point of the graph of $y = (x - 1)(x + 2)$

Solution



Due to the symmetry of the curve about the vertical line through the turning point, the x-coordinate of the turning point will be $\frac{1}{2}(-2+1)=-\frac{1}{2}$

Then the y-coordinate is $=\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}+2\right)=\left(\frac{-3}{2}\right)\left(\frac{3}{2}\right)=-\frac{9}{4}$

Alternatively, we can complete the square:

$(x-1)(x+2)=x^2+x-2=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-2=\left(x+\frac{1}{2}\right)^2-\frac{9}{4}$

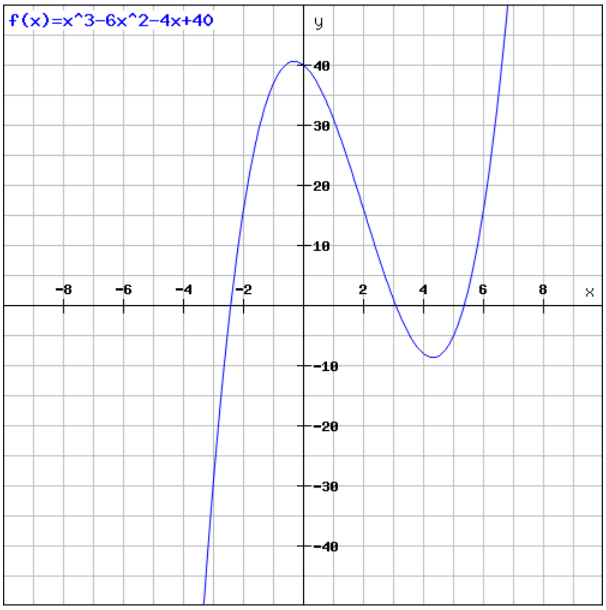
giving the turning point of $\left(-\frac{1}{2},-\frac{9}{4}\right)$

Point of Inflexion

Turning point of the gradient

Necessary & sufficient condition:

$\frac{d^2y}{dx^2} = 0$ and 1st non-zero derivative (excluding $\frac{dy}{dx}$) is of odd order



Find the x-coordinate of the point of inflexion of the cubic

$y = ax^3 + bx^2 + cx + d$

Solution

For $f(x) = ax^3 + bx^2 + cx + d$,

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f'''(x) = 6a \neq 0$

So point of inflexion of the cubic occurs when $f''(x) = 0$

$\Rightarrow x = -\frac{b}{3a}$

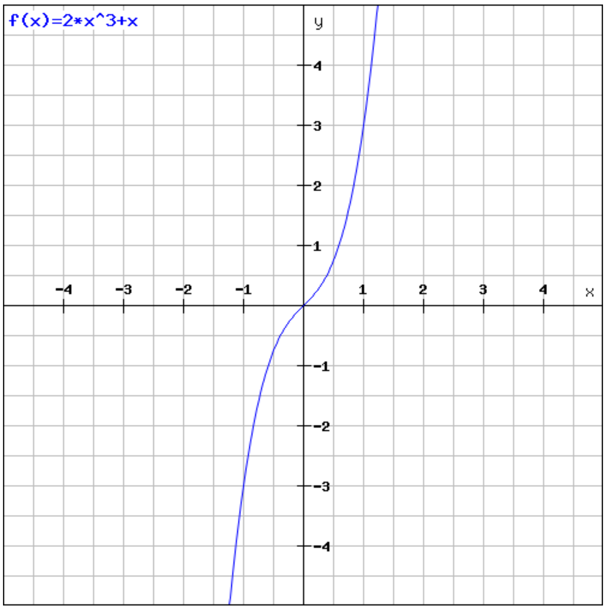
Sketch (i) $y = 2x^3 + x$, and (ii) $y = 2x^3 - x$

Solution

(i) $y = 2x^3 + x = x(2x^2 + 1)$; so exactly one real root

$\frac{dy}{dx} = 6x^2 + 1 > 0$

$\frac{d^2y}{dx^2} = 12x$; so $\frac{d^2y}{dx^2} = 0$ when $x = 0$



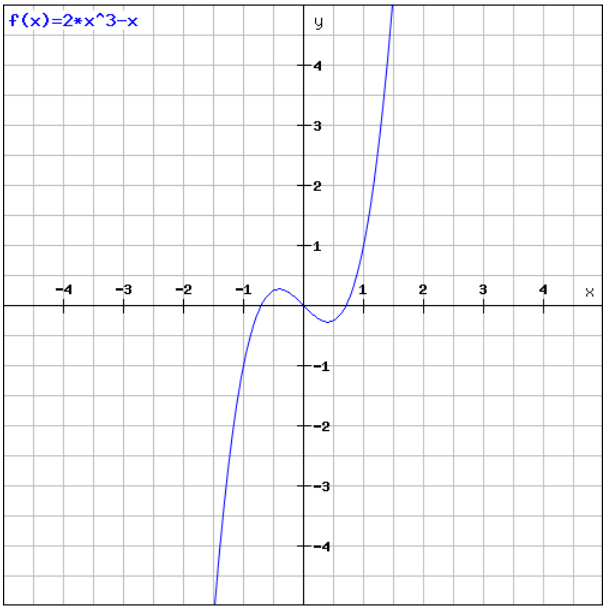
(ii) $y = 2x^3 - x = x(2x^2 - 1) = x(\sqrt{2}.x - 1)(\sqrt{2}\,.x + 1)$;

so 3 real roots

$\frac{dy}{dx} = 6x^2 - 1$

$\frac{d^2y}{dx^2} = 12x$; so $\frac{d^2y}{dx^2} = 0$ when $x = 0$, and then $\frac{dy}{dx} = -1$

Also, $\frac{dy}{dx} = 0$ when $x = \pm\frac{1}{\sqrt{6}}$



Why do cubic functions have rotational symmetry (of order 2) about the point of inflexion?

[Why do cubic functions have rotational symmetry (of order 2) about the point of inflexion?]

Hint: WLOG, consider a cubic that passes through the Origin.

[Why do cubic functions have rotational symmetry (of order 2) about the point of inflexion?]

Consider a cubic of the form $y = ax^3 + bx^2 + cx + d$

WLOG translate it so that its point of inflexion is the Origin.

Then $y = f(x) = ax^3 + cx$ (as the PoI is at $x = -\frac{b}{3a}$)

And $f(-x) = -f(x)$

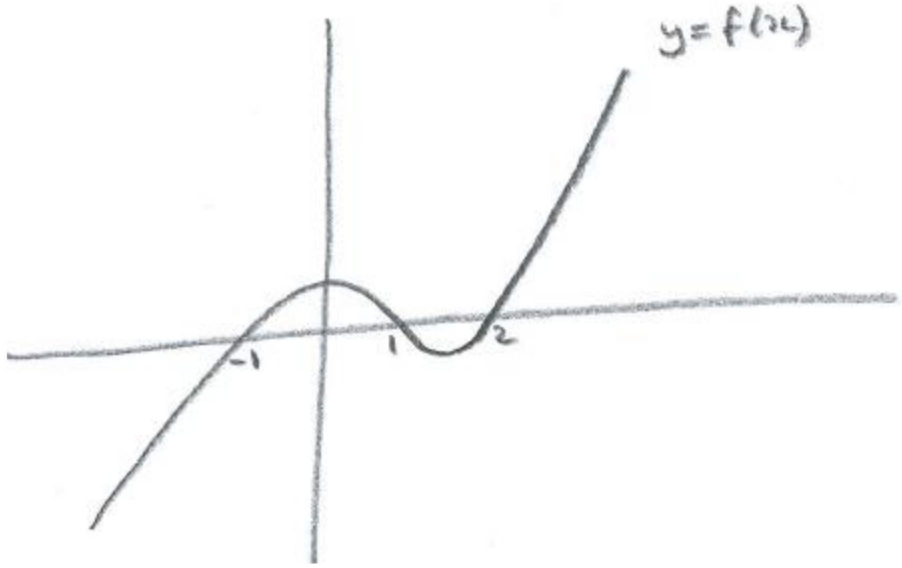
Rotational symmetry ⇒ point of inflexion lies midway between any turning points.

If $f(x) = (x+1)(x-1)(x-2)$, sketch the following:

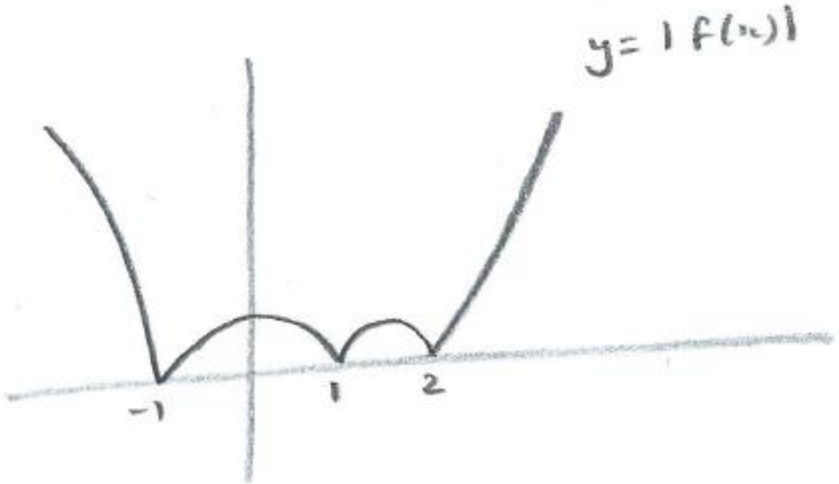
(i) $y = f(x)$ (ii) $y = |f(x)|$ (iii) $y = f(|x|)$ (iv) $|y| = f(x)$

Solution

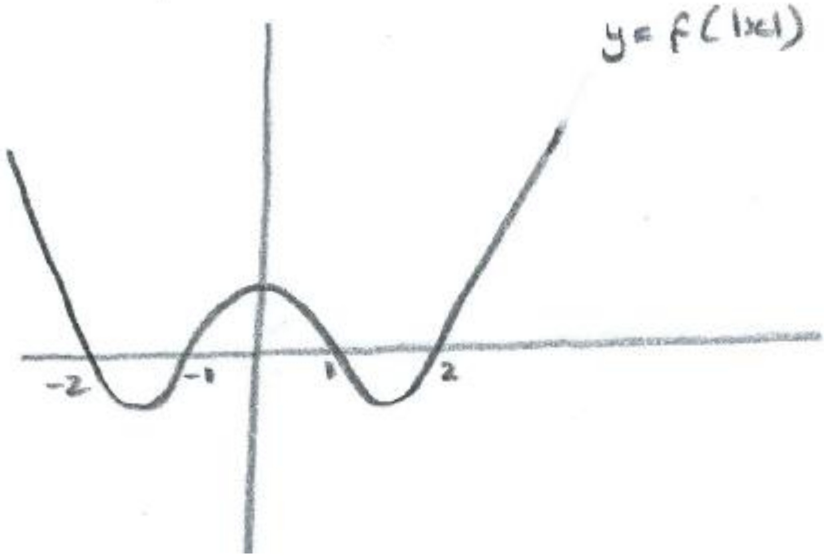
(i) $y = f(x)$

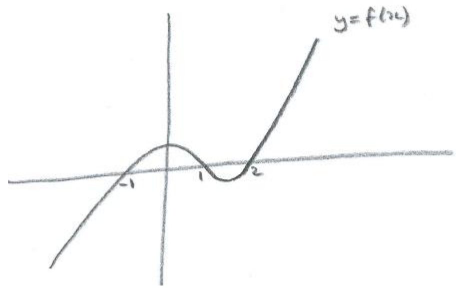


(ii) $y = |f(x)|$

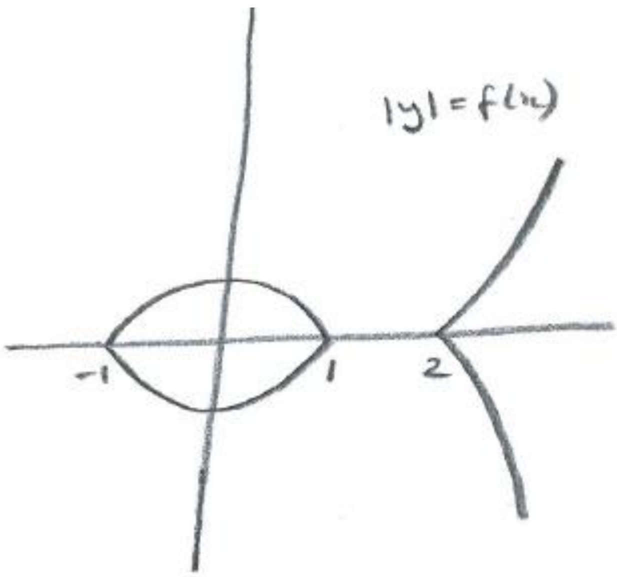


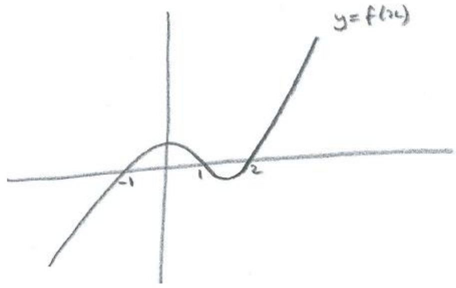
(iii) $y = f(|x|)$





(iv) $|y| = f(x)$

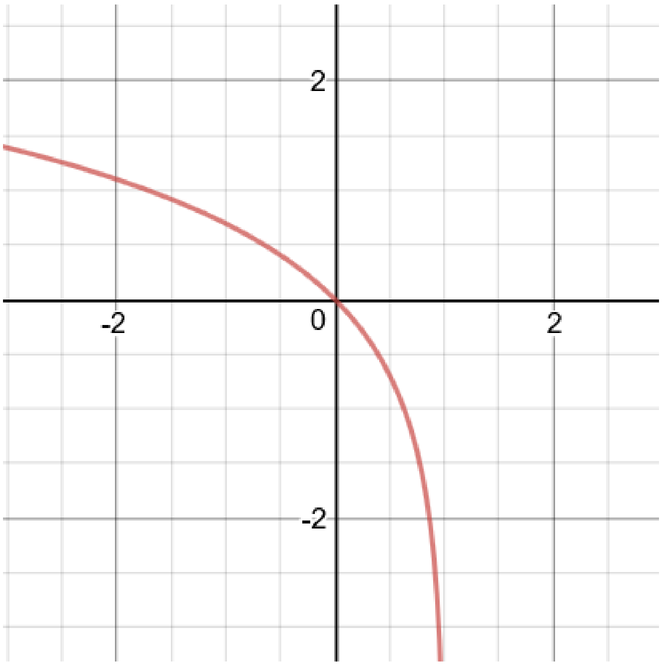




Sketch $y = ln\,(1 - x)$

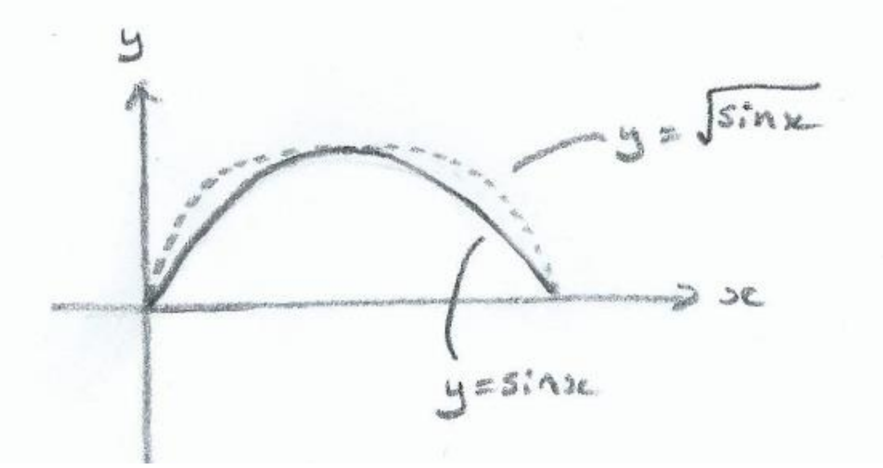
Solution

$y = \ln(1-x)$ is the reflection in $x = \frac{1}{2}$ of $y = lnx$



Sketch (i) $y = \sqrt{sinx}$ and (ii) $y = (sinx)^{\frac{1}{n}}$ for large positive integer n (for $0 \leq x \leq \pi$ in both cases).

Solution



(i) Note that, for $0 < y < 1, \sqrt{y} > y$

So, for $y = \sqrt{sinx}$, the graph will hug the $y - axis$ more than for $y = sinx$.

Also, if $f(x) = \sqrt{sinx}$, $f'(x) = \frac{1}{2}(sinx)^{-\frac{1}{2}}cosx$,

so that $f'(0) = \infty$ (strictly speaking, it is 'undefined');

ie the graph is vertical at $x = 0$ (and also $x = \pi$, by symmetry).

(ii) The effect is greater for larger n, and the graph tends to a rectangular shape.

Sketch $y = \frac{x}{\sqrt{x^2+p}}$, where p is a positive constant, for $x \geq 0$

Solution

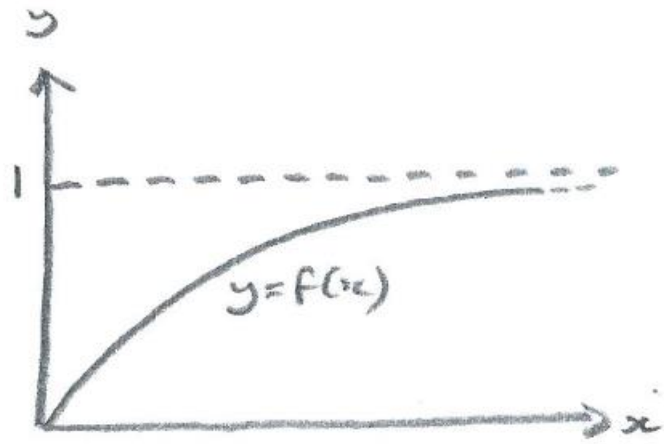
Writing $f(x) = \frac{x}{\sqrt{x^2+p}}$,

$f(0) = 0$ and $f(x) \to 1^-$ as $x \to \infty$

$$f(x) = \frac{x}{\sqrt{x^2+p}} \Rightarrow f'(x) = \frac{\sqrt{x^2+p} - x.\frac{1}{2}(x^2+p)^{-\frac{1}{2}}.2x}{x^2+p}$$

$$= \frac{(x^2+p)-x^2}{(x^2+p)^{\frac{3}{2}}} = \frac{p}{(x^2+p)^{\frac{3}{2}}} > 0 \text{ for } x \geq 0$$

And $f''(x) = p\left(-\frac{3}{2}\right)(x^2+p)^{-\frac{5}{2}}(2x) < 0$ for $x > 0$



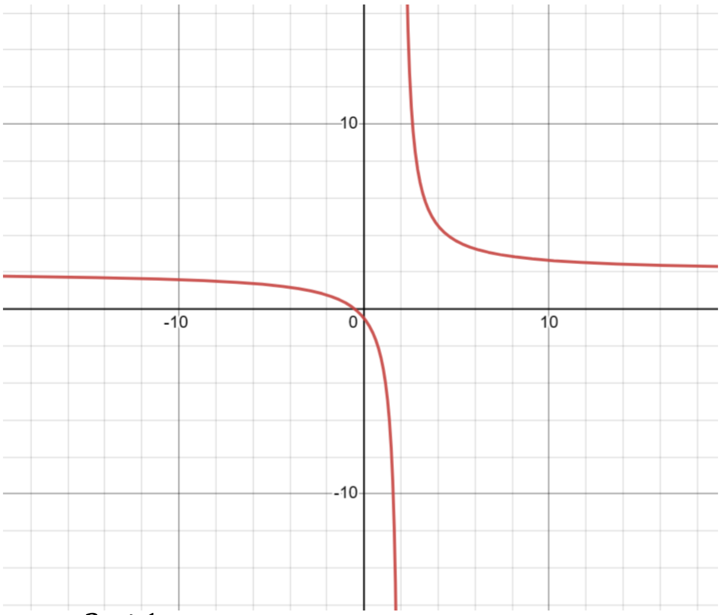
Checklist of curve sketching devices

(i) Intercepts with axes

(ii) Behaviour for large positive and negative x (and y)

(iii) Vertical and horizontal asymptotes

Sketch $y = \frac{2x+1}{x-2}$



$y = \frac{2x+1}{x-2}$

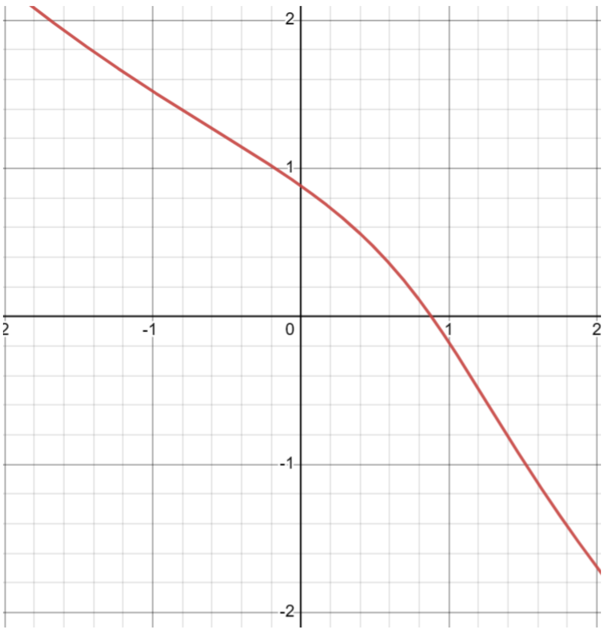
(iv) Symmetries:

(a) about $x = a$ (special case: $x = 0$; ie y-axis)

(b) rotational symmetry (odd function)

(c) symmetry about $y = x$

eg $sinhx + sinhy = 1$



Consider domain (line of symmetry may lie mid-way between limits of domain). [See STEP 2011, P2, Q1]

(v) Gradient of function

(vi) Greatest or least value of a function

- but stationary points only indicate local maxima and minima
- a greatest or least value may occur at a boundary of the domain

Examples where $f(x) \geq 0$:

(i) $f(x) = [g(x)]^2 + [h(x)]^2$

(ii) For $x \geq a$: establish that $f(a) \geq 0$ and that $f'(x) \geq 0$ for $x \geq a$.

(iii) $f(x) = xsinhx[g(x)]^2$ (as x & $sinhx$ will always have the same sign - unless they are both zero)

(vii) Points of inflexion

(viii) Transformation of a simpler function

(ix) Breaking down the domain