STEP/Counting Q8 (11/6/23)

A 4-digit password is made up of numbers from 0 to 4, where the numbers can be repeated, but have to be ordered from largest to smallest. Show that there are 70 possible passwords.

Solution

Consider a simpler version of the problem, with just 3 numbers, each of which can be $0, 1\ or\ 2$

Then the following are possible:

$222, 221, 220, 211, 210, 200$

$111, 110, 100$

000

It may be possible to come up with a recurrence relation of some sort. Let $f(m,n)$ be the number of possibilities when we can choose between $0, 1, \dots, m$ for each digit, and there are n digits in the password.

From the above, we have $f(2,3) = f(2,2) + f(1,2) + f(0,2)$

and this can be generalised to

$f(m,n) = f(m,n-1) + f(m-1,n-1) + \cdots + f(0,n-1)$

We also note that $f(m,1) = m + 1$

Applying this to the problem in question,

$f(4,4) = f(4,3) + f(3,3) + f(2,3) + f(1,3) + f(0,3)$

$= [f(4,2) + f(3,2) + \cdots + f(0,2)]$

$+[f(3,2) + f(2,2) + \cdots + f(0,2)]$

$+ \cdots + f(0,2)$

$= f(4,2) + 2f(3,2) + 3f(2,2) + \cdots + 5f(0,2)$

$= [f(4,1) + f(3,1) + \cdots + f(0,1)]$

$+2[f(3,1) + f(2,1) + \cdots + f(0,1)]$

$+ \cdots 5f(0,1)$

$= f(4,1) + (1+2)f(3,1) + (1+2+3)f(2,1)$

$+(1+2+3+4)f(1,1) + (1+2+3+4+5)f(0,1)$

$= 5 + 3(4) + 6(3) + 10(2) + 15(1)$

$= 5 + 12 + 18 + 20 + 15 = 70$

[This method can be applied to the original BMO problem, though it would become unwieldy if the length of the password exceeded 6 digits.]

Alternative (quicker) method [based on the official solutions, contained in "A Mathematical Olympiad Primer" by Geoff Smith]:

Each possibility can be represented using the following system:

3220 is represented by DXDXXDDX

and 4311 is represented by $XDXDDXXD$

For 3220, the 1st letter D means that we are dropping by 1 from the maximum of 4, and the 2nd letter X means that we have reached the value of the 1st digit; similarly the next 2 letters DX mean that we are dropping by another 1 to arrive at the 2nd digit; the 5th letter X means that we don't drop at all to arrive at the 3rd digit; the final letters DDX mean that we drop by 2 to arrive at the last digit.

For 4311, we need a D on the end to get down to 0.

In all cases there will be 4 Ds and 4 Xs, and the Ds and Xs can appear in any of the places, so that the number of possibilities is $\binom{8}{4} = 70$.

[See the official solutions for a couple of other approaches.]