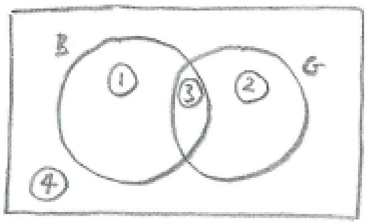
STEP/Counting Q5 (11/6/23)

2 boys and 3 girls are to sit in a row. How many arrangements are there in which the 2 boys are not next to each and the 3 girls are also not next to each other?

Solution

A Venn diagram could be created to illustrate the various possibilities.

Let B denote the cases where the boys are together, and G the cases where the girls are together. There are then 4 regions in the Venn diagram, as shown below.



We are interested in region (4).

The total number of ways of arranging the 5 children is $5! = 120$. This is (1)+(2)+(3)+(4).

To find (3), where the boys are together and also the girls: Let M denote the block of boys, and F the block of girls [to avoid confusion with the B and G already used in the Venn diagram].

We can then have either MF or FM. Multiplying this figure of 2 by 2!3! in order to allow for the arrangements of the boys within M etc, gives (3)=24.

To find B = (1)+(3): consider the number of arrangements of $MF_1F_2F_3$

This gives 4! (ways of arranging these 4 items) × 2! (ways of arrangements the boys within their block); ie B=(1)+(3)=48

Similarly, $G = (2) + (3) =$ number of arrangements of $FM_1M_2 = 3!\,3! = 36$

To summarise so far,

$(1) + (2) + (3) + (4) = 120$

$(3) = 24$

$(1) + (3) = 48$

$(2) + (3) = 36$

Hence $(1) = 24, (2) = 12$ and $(4) = 120 - 24 - 12 - 24 = 60$

[We could also list the possible arrangements in the situation where the boys and girls are taken to be indistinguishable, and multiply by $2!\,3! = 12$ in each case (the number of ways of arranging the boys and girls for each of these possibilities).

Labelling each with the appropriate region from the Venn diagram:

MMFFF (3)

MFMFF (4)

MFFMF (4)

MFFFM (2)

FMMFF (1)

FMFMF (4)

FMFFM (4)

FFMMF (1)

FFMFM (4)

FFFMM (3)

This gives (1)=2, (2)=1, (3)=2, (4)=5

And multiplying each by 12 gives the figures arrived at previously.]