STEP/Counting Q10 (11/7/23)

(i) How many ways are there of writing the pair of integers (i, j), given that $1 \leq i < j \leq n$?

(ii) Show that there are $n(n-1)(n-2)$ ways of writing the integers (i, j, k, l), given that $1 \leq i < j \leq n$ and

$1 \leq k < l \leq n$, and such that exactly one of the numbers is

repeated (eg (1,2,1,3) or (1,2,2,3), but not (1,2,1,2))

Solution

(i) Method 1

$^{n}C_2 = \frac{1}{2}n(n-1)$

Method 2

$i = 1$: $n-1$ possible values for j

$i = 2$: $n-2$ possible values for j

... $i = n-1$: 1 possible value for j

giving a total of $(n-1) + (n-2) + \cdots 1 = \frac{1}{2}(n-1)n$

(ii) **Method 1**

Category A: The repeated numbers are i & k (eg (4,7,4,9) or (4,9,4,7))

Category B_1: The repeated numbers are i & l (eg (4,7,2,4)

Category B_2: The repeated numbers are j & k (eg (2,4,4,7))

Category C: The repeated numbers are j & l (eg (1,4,2,4) or (2,4,1,4))

To count the number of items in Category A; initially with $j < l$:

For items of the form $(1, j, 1, l)$:

$^{n-1}C_2 = \frac{1}{2}(n-1)(n-2)$

For items of the form $(2, j, 2, l)$:

$^{n-2}C_2 = \frac{1}{2}(n-2)(n-3)$

... For items of the form $(n-2, j, n-2, l)$: 1

So total number of items in Category A, allowing for both $j < l$ and $l < j$ is $2\sum_{r=1}^{n-2}\frac{1}{2}(n-r)(n-r-1)$

Writing $k = n-r-1$, this becomes

$\sum_{k=n-2}^{1} k(k-1)$

or $\sum_{k=1}^{n-2} k(k-1) = \frac{1}{6}(n-2)(n-1)(2n-3) + \frac{1}{2}(n-2)(n-1)$

$= \frac{1}{6}(n-1)(n-2)[2n-3+3] = \frac{n}{3}(n-1)(n-2)$

To count the number of items in Category B_1:

Items of the form $(2, j, k, 2)$: $(n-2).1$

Items of the form $(3, j, k, 3)$: $(n-3).2$

... Items of the form $(n-1, j, k, n-1)$: $1.(n-2)$

And the number of items in Category B_2 will be the same (as (2,4,4,7) corresponds to (4,7,2,4) etc).

So total number of items in Category B is

$2\sum_{r=1}^{n-2} r(n-r-1) = 2(n-1).\frac{1}{2}(n-2)(n-1)$

$-\frac{2}{6}(n-2)(n-1)(2n-3)$

$= \frac{1}{3}(n-1)(n-2)[3n-3-(2n-3)]$

$= \frac{n}{3}(n-1)(n-2)$

To count the number of items in Category C:

This will be the same as for Category A, with ${}^{n-1}C_2$ items of the form (i, r, k, r).

Thus the total number of items in Categories A, B & C (together) is

$3 \times \frac{n}{3}(n-1)(n-2) = n(n-1)(n-2)$

[eg for $n = 3, n(n-1)(n-2) = 6$:

(1,2)(1,3) , (1,2)(2,3) , (1,3)(1,2) , (1,3)(2,3) , (2,3)(1,2) , (2,3)(1,3)]

Method 2

For each choice of repeated number, there are ${}^{n-1}C_2$ ways of choosing the other two numbers, and in each case exactly one of the categories A, B or C must occur: Suppose, for example, that the repeated number is 4. Then the other two numbers will either both be less than 4, or both be greater than 4, or lie on either side of 4 (corresponding to the categories C, A & B). But for each case there are two possibilities: eg (4,7)(4,8) or (4,8)(4,7)

So the total number of cases is $n \times 2 \times {}^{n-1}C_2 = 2n \,.\, \frac{(n-1)(n-2)}{2}$

$= n(n-1)(n-2)$

[Thus the categories A, B & C are useful for the purpose of establishing all the possible cases, but there is a simpler way of counting the cases.]

Method 3

There are $({}^nC_2)^2$ ways of writing (i, j, k, l), if $1 \leq i < j \leq n$ and

$1 \leq k < l \leq n$, but any possibilities are allowed; ie including
P: (1,2)(1,2) [where two distinct numbers are repeated] and
Q: (1,2)(3,4) [where no numbers are repeated]

The number of P cases is ${}^nC_2$, and the number of Q cases is:

${}^nC_2 \times {}^{n-2}C_2$

Hence required number is:

$({}^nC_2)^2 - {}^nC_2 - {}^nC_2 \times {}^{n-2}C_2$

$= {}^nC_2({}^nC_2 - 1 - {}^{n-2}C_2)$

$= \frac{n(n-1)}{2}(\frac{n(n-1)}{2} - 1 - \frac{(n-2)(n-3)}{2})$

$= \frac{n(n-1)}{4}(n^2 - n - 2 - (n^2 - 5n + 6))$

$= \frac{n(n-1)}{4}(4n - 8)$

$= n(n-1)(n-2)$