STEP - Complex Number Methods (9 pages; 28/02/15)

Introduction

Complex numbers can be treated as either points in the Argand diagram, translations in the Argand diagram (as for vectors), spiral enlargements (in the Argand diagram), or just numbers (without reference to the Argand diagram).

Methods

(1) Substitutions for z

(a) Let $z = a + bi$ [see Examples A, B, C]

(b) Let $z = r(cos\theta + isin\theta)$ [see Examples D, J, K]

(c) Let $z = re^{i\theta}$ [see Example D]

(2) Equating real and imaginary parts [see Examples A, B, C]

(3) Vector-style translations [see Examples E, G, H, I, K, L]

(4) Spiral enlargements [see Examples H, I, J, K, L]

(5) Number interpretations [see Examples A, C, E]

(6) de Moivre's theorem [see Examples D, E]

(7) $arg(u) = arg(v) \Rightarrow u = kv$, where k is real,

since $arg(u) - arg(v) = 0 \Rightarrow arg\left(\frac{u}{v}\right) = 0 \Rightarrow \frac{u}{v} = k$

(or just from the fact that u and v have the same direction)

(8) Operations with conjugates [see Example F]

$(u + v)^* = u^* + v^*$; $(uv)^* = u^*v^*$; $uu^* = |u|^2$

(9) 'Sum of two squares' [see Example C]

(10) Finding complex roots of equations:

(i) Factor theorem

(ii)Relations between roots of polynomial equations ($\sum \alpha = -\frac{b}{a}$ etc) also apply when roots are complex (and the coefficients of the equation can be complex)

(iii) Complex roots of polynomials **with real coefficients** occur in conjugate pairs.

Examples

(A) To solve $(2+i)z+3=0$

Method 1: $z=-\frac{3}{2+i}=-\frac{3(2-i)}{(2+i)(2-i)}$ etc

Method 2: Let $z=a+bi$; then equate real and imaginary parts

(B) To find $\sqrt{24-10i}$, let $24-10i=(a+bi)^2$; then equate real and imaginary parts

(C) To solve $z^2-2z+2=0$

Method 1: Quadratic formula

Method 2: Let $z=a+bi$; then equate real and imaginary parts

Method 3: $z^2-2z+2=0 \Rightarrow (z-1)^2+1^2=0$

$\Rightarrow ([z-1]+i)([z-1]-i)=0 \Rightarrow z=1-i \ or \ 1+i$

(D) To find $(1+i)^4$

$1+i=\sqrt{2}e^{i\pi/4}$

Hence $(1+i)^4=4e^{i\pi}=4(cos\pi+isin\pi)=-4$

(E) Roots of $z^5=1$

If $\omega=cos\left(\frac{2\pi}{5}\right)+jsin\left(\frac{2\pi}{5}\right)$, then $1+\omega+\omega^2+\omega^3+\omega^4=0$

either by considering the vector representations in Figures 1 and 2

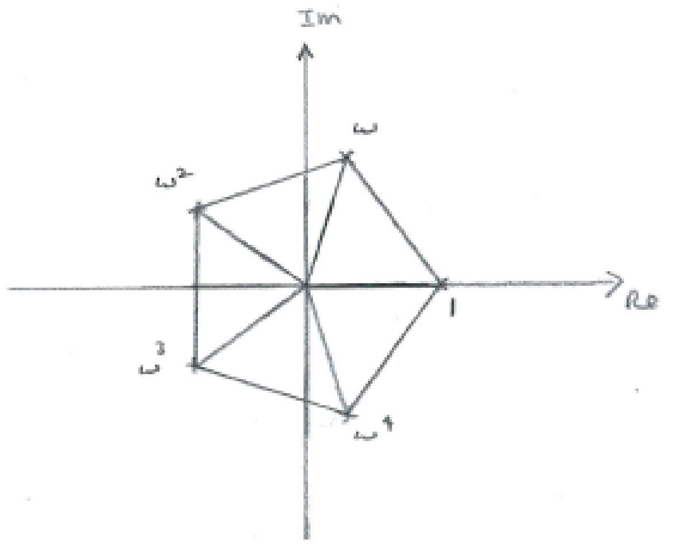

Figure 1

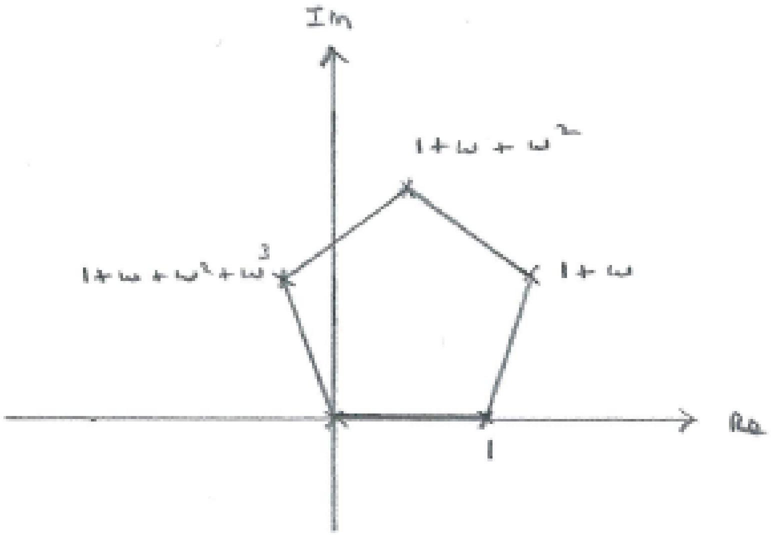

Figure 2

or from the sum of a geometric series: $\frac{1(\omega^5-1)}{\omega-1} = 0$, since $\omega^5 = 1$

(F) $p - q = ki(r+s) \Rightarrow p^* - q^* = k(-i)(r^* + s^*)$, where k is real

(G) Referring to Figure 3, the mid-point M of the line segment AB in the Argand diagram is represented by the complex number $m = \frac{1}{2}(a + b)$, where A and B are represented by a and b

respectively (since $m - a = b - m$)

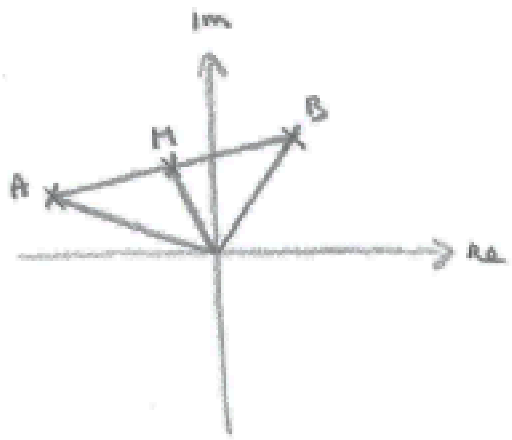

Figure 3

(H) Referring to Figure 4, if the corners of a right-angled triangle are the points P, Q & R in the Argand diagram (in anti-clockwise order, with the right-angle being at Q), represented by the complex numbers p, q & r, then $p - q = ki(r - q)$, where k is real

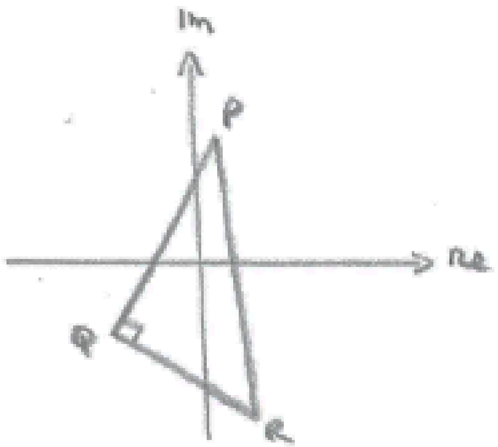

Figure 4

(I) Referring to Figure 5, let A be the point $-1 - 3i$ and B be the point $-4 + 4i$. ABC is a right-angled isosceles triangle, as shown. To find the point C:

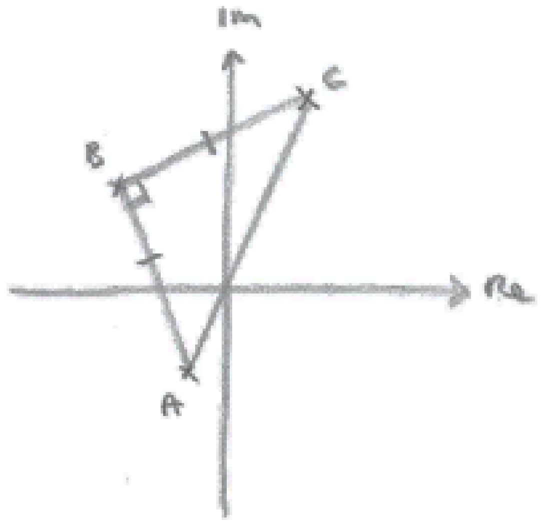

Figure 5

Method 1

$\overrightarrow{AB} = (-4 + 4i) - (-1 - 3i) = -3 + 7i$

$\overrightarrow{AC}$ is obtained from $\overrightarrow{AB}$ by a rotation of $-\frac{\pi}{4}$ and an enlargement of scale factor $\sqrt{2}$

Hence $\overrightarrow{AC} = (-3 + 7i)\sqrt{2}(\cos\left(-\frac{\pi}{4}\right) + isin\ \left(-\frac{\pi}{4}\right))$

$= (-3 + 7i)\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = (-3 + 7i)(1 - i)$

$= -3 + 7i + 3i + 7 = 4 + 10i$

Hence $C = A + \overrightarrow{AC} = (-1 - 3i) + (4 + 10i) = 3 + 7i$

Method 2

$\overrightarrow{BA} = (-1-3i) - (-4+4i) = 3 - 7i$

$\overrightarrow{BC} = i\overrightarrow{BA} = 7 + 3i$

Hence $C = B + \overrightarrow{BC} = (-4+4i) + (7+3i) = 3 + 7i$

(J) In general, referring to Figure 6, if points A and B in the Argand diagram are represented by the complex numbers a and b, then we can write $b = sa$, where $s = |s|(cos\theta + isin\theta)$ represents a rotation through angle θ, together with an enlargement of scale factor $|s|$.

Triangle OAB is then similar to triangle OIS in Figure 7, where S is the complex number s, and I is the real number 1.

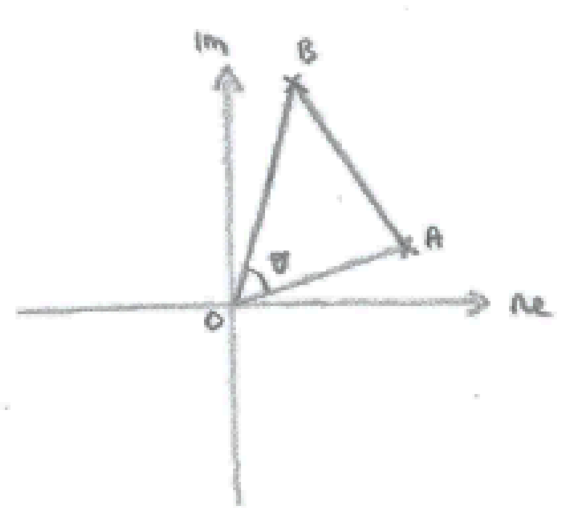

Figure 6

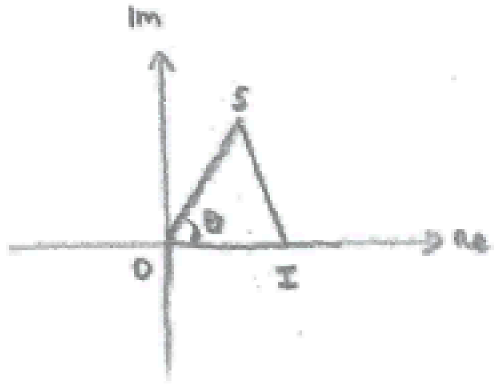

Figure 7

(K) Referring to Figure 8, let A, B & C be the corners of an equilateral triangle in the Argand diagram (in anti-clockwise order), represented by the complex numbers a, b & c

Then $a - b = (c - b)(\cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right))$

[since $|a - b| = |c - b|$]

$= (c - b)(\frac{1}{2} + i\frac{\sqrt{3}}{2})$

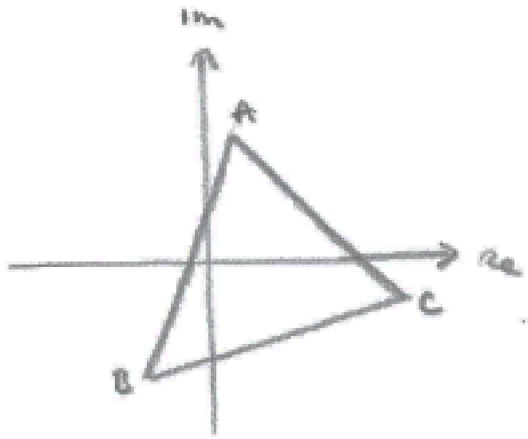

Figure 8

(L) Referring to Figure 9, let A, B & C be the corners of an isosceles triangle in the Argand diagram (in anti-clockwise order, with equal angles at B & C), represented by the complex numbers a, b & c

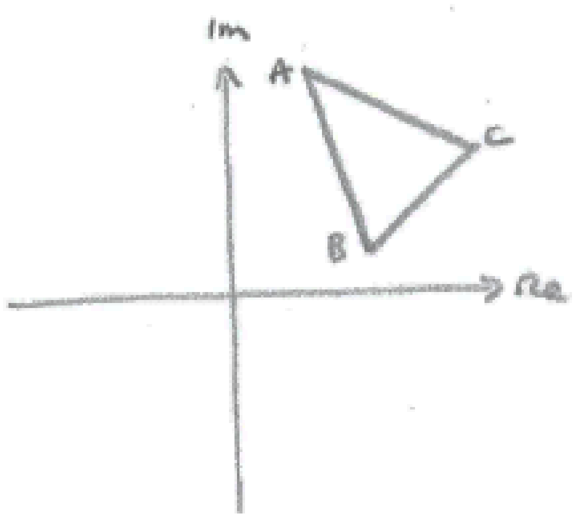

Figure 9

Then $arg\left(\frac{a-b}{c-b}\right) = arg\left(\frac{b-c}{a-c}\right)$

$\Rightarrow \frac{a-b}{c-b} = k(\frac{b-c}{a-c})$

If the triangle is equilateral, $k = 1$

and $(a-b)(a-c) = -(b-c)^2$

$\Rightarrow a^2 - ba - ac + bc = -b^2 - c^2 + 2bc$

$\Rightarrow a^2 + b^2 + c^2 = ab + ac + bc$