

**STEP/Algebra Q5 (13/6/23)**

(i) Find an expansion for  $(a + b + c)^3$ , and give a justification for the coefficients.

(ii) Extend this to  $(a + b + c)^4$

**Solution**

(i) By an ordinary expansion:

$$\begin{aligned}
 (a + b + c)^3 &= ([a + b] + c)^3 \\
 &= (a + b)^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3 \\
 &= (a^3 + 3a^2b + 3ab^2 + b^3) + (3a^2c + 3b^2c + 6abc) \\
 &\quad + (3ac^2 + 3bc^2) + c^3 \\
 &= (a^3 + b^3 + c^3) + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) \\
 &\quad + 6abc
 \end{aligned}$$

Alternatively, this could have been deduced by noting that the terms fall into one of the 3 groups above.

Then there is only 1 way of creating an  $a^3$  term from

$(a + b + c)(a + b + c)(a + b + c)$ ; namely by choosing the  $a$  from each of the 3 brackets.

There are 3 ways of creating an  $a^2b$  term: 3[number of ways of choosing the  $b$ ]  $\times$  1[number of ways of choosing two  $a$ s from the remaining 2 brackets].

Finally, there are 6 ways of creating an  $abc$  term: 3[number of ways of choosing the  $a$ ]  $\times$  2[number of ways of choosing the  $b$  from the remaining 2 brackets]  $\times$  1[number of ways of choosing the  $c$  from the remaining bracket].

The final expression then follows by symmetry.

$$\begin{aligned}
 \text{(ii) } (a + b + c)^4 &= (a^4 + b^4 + c^4) \\
 &\quad + 4(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b) \\
 &\quad + 6(a^2b^2 + a^2c^2 + b^2c^2) + 12(a^2bc + b^2ac + c^2ab)
 \end{aligned}$$

For the  $a^2b^2$  term etc, there are  $\binom{4}{2} = 6$  ways of choosing the brackets from  $(a + b + c)(a + b + c)(a + b + c)(a + b + c)$  to give  $a^2$ , and then just 1 way of obtaining the  $b^2$  term.

For the  $a^2bc$  term etc, there are  $\binom{4}{2} = 6$  ways of choosing the brackets for the  $a^2$  again, multiplied by the 2 ways of choosing brackets for the  $b$  and  $c$ .

For further investigation: the 'trinomial' expansion of

$(a + b + c)^n$  can be shown to be  $\sum_{(i+j+k=n)} \binom{n}{i,j,k} a^i b^j c^k$ ,

where  $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

(with a further extension to the 'multinomial' expansion of

$(a_1 + a_2 + \dots + a_m)^n$ )