Section A: Pure Mathematics

1 A *proper factor* of an integer N is a positive integer, not 1 or N, that divides N.

(i) Show that $3^2 \times 5^3$ has exactly 10 proper factors. Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors.

(ii) Let N be the smallest positive integer that has exactly 426 proper factors. Determine N, giving your answer in terms of its prime factors.

2 A curve has the equation

$$y^3 = x^3 + a^3 + b^3\,,$$

where a and b are positive constants. Show that the tangent to the curve at the point $(-a, b)$ is

$$b^2y - a^2x = a^3 + b^3\,.$$

In the case $a = 1$ and $b = 2$, show that the x-coordinates of the points where the tangent meets the curve satisfy

$$7x^3 - 3x^2 - 27x - 17 = 0\,.$$

Hence find positive integers p, q, r and s such that

$$p^3 = q^3 + r^3 + s^3\,.$$

3 **(i)** By considering the equation $x^2 + x - a = 0$, show that the equation $x = (a - x)^{\frac{1}{2}}$ has one real solution when $a \geqslant 0$ and no real solutions when $a < 0$.

Find the number of distinct real solutions of the equation

$$x = \big((1 + a)x - a\big)^{\frac{1}{3}}$$

in the cases that arise according to the value of a.

(ii) Find the number of distinct real solutions of the equation

$$x = (b + x)^{\frac{1}{2}}$$

in the cases that arise according to the value of b.

4 The sides of a triangle have lengths $p-q$, p and $p+q$, where $p>q>0$. The largest and smallest angles of the triangle are α and β, respectively. Show by means of the cosine rule that

$$4(1-\cos\alpha)(1-\cos\beta)=\cos\alpha+\cos\beta\,.$$

In the case $\alpha=2\beta$, show that $\cos\beta=\frac{3}{4}$ and hence find the ratio of the lengths of the sides of the triangle.

5 A right circular cone has base radius r, height h and slant length ℓ. Its volume V, and the area A of its curved surface, are given by

$$V=\tfrac{1}{3}\pi r^2 h\,,\qquad A=\pi r\ell\,.$$

(i) Given that A is fixed and r is chosen so that V is at its stationary value, show that $A^2=3\pi^2 r^4$ and that $\ell=\sqrt{3}\,r$.

(ii) Given, instead, that V is fixed and r is chosen so that A is at its stationary value, find h in terms of r.

6 **(i)** Show that, for $m>0$,

$$\int_{1/m}^{m}\frac{x^2}{x+1}\,\mathrm{d}x=\frac{(m-1)^3(m+1)}{2m^2}+\ln m\,.$$

(ii) Show by means of a substitution that

$$\int_{1/m}^{m}\frac{1}{x^n(x+1)}\,\mathrm{d}x=\int_{1/m}^{m}\frac{u^{n-1}}{u+1}\,\mathrm{d}u\,.$$

(iii) Evaluate:

(a) $$\int_{1/2}^{2}\frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x\;;$$

(b) $$\int_{1}^{2}\frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x\,.$$

7 Show that, for any integer m,

$$\int_0^{2\pi} \mathrm{e}^x \cos mx \, \mathrm{d}x = \frac{1}{m^2+1}\big(\mathrm{e}^{2\pi} - 1\big)\,.$$

(i) Expand $\cos(A+B) + \cos(A-B)$. Hence show that

$$\int_0^{2\pi} \mathrm{e}^x \cos x \cos 6x \, \mathrm{d}x \;=\; \tfrac{19}{650}\big(\mathrm{e}^{2\pi} - 1\big)\,.$$

(ii) Evaluate $\displaystyle\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x \, \mathrm{d}x\,.$

8 **(i)** The equation of the circle C is

$$(x-2t)^2 + (y-t)^2 = t^2,$$

where t is a positive number. Show that C touches the line $y = 0\,.$

Let α be the acute angle between the x-axis and the line joining the origin to the centre of C. Show that $\tan 2\alpha = \frac{4}{3}$ and deduce that C touches the line $3y = 4x\,.$

(ii) Find the equation of the incircle of the triangle formed by the lines $y = 0$, $3y = 4x$ and $4y + 3x = 15\,.$

Note: The *incircle* of a triangle is the circle, lying totally inside the triangle, that touches all three sides.

Section B: Mechanics

9 Two particles P and Q are projected simultaneously from points O and D, respectively, where D is a distance d directly above O. The initial speed of P is V and its angle of projection *above* the horizontal is α. The initial speed of Q is kV, where $k > 1$, and its angle of projection *below* the horizontal is β. The particles collide at time T after projection.

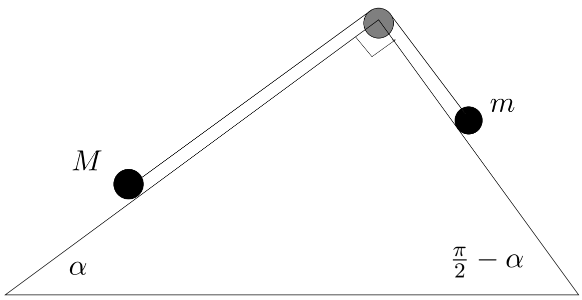
Show that $\cos\alpha = k\cos\beta$ and that T satisfies the equation

$$(k^2 - 1)V^2T^2 + 2dVT\sin\alpha - d^2 = 0\,.$$

Given that the particles collide when P reaches its maximum height, find an expression for $\sin^2\alpha$ in terms of g, d, k and V, and deduce that

$$gd \leqslant (1+k)V^2\,.$$

10 A triangular wedge is fixed to a horizontal surface. The base angles of the wedge are α and $\frac{\pi}{2} - \alpha$. Two particles, of masses M and m, lie on different faces of the wedge, and are connected by a light inextensible string which passes over a smooth pulley at the apex of the wedge, as shown in the diagram. The contacts between the particles and the wedge are smooth.



(i) Show that if $\tan\alpha > \dfrac{m}{M}$ the particle of mass M will slide down the face of the wedge.

(ii) Given that $\tan\alpha = \dfrac{2m}{M}$, show that the magnitude of the acceleration of the particles is

$$\frac{g\sin\alpha}{\tan\alpha + 2}$$

and that this is maximised at $4m^3 = M^3$.

11 Two particles move on a smooth horizontal table and collide. The masses of the particles are m and M. Their velocities before the collision are $u\mathbf{i}$ and $v\mathbf{i}$, respectively, where $\mathbf{i}$ is a unit vector and $u > v$. Their velocities after the collision are $p\mathbf{i}$ and $q\mathbf{i}$, respectively. The coefficient of restitution between the two particles is e, where $e < 1$.

(i) Show that the loss of kinetic energy due to the collision is

$$\tfrac{1}{2}m(u-p)(u-v)(1-e)\,,$$

and deduce that $u \geqslant p$.

(ii) Given that each particle loses the same (non-zero) amount of kinetic energy in the collision, show that

$$u+v+p+q=0\,,$$

and that, if $m \neq M$,

$$e = \frac{(M+3m)u+(3M+m)v}{(M-m)(u-v)}\,.$$

Section C: Probability and Statistics

12 Prove that, for any real numbers x and y, $x^2 + y^2 \geqslant 2xy$.

(i) Carol has two bags of sweets. The first bag contains a red sweets and b blue sweets, whereas the second bag contains b red sweets and a blue sweets, where a and b are positive integers. Carol shakes the bags and picks one sweet from each bag without looking. Prove that the probability that the sweets are of the same colour cannot exceed the probability that they are of different colours.

(ii) Simon has three bags of sweets. The first bag contains a red sweets, b white sweets and c yellow sweets, where a, b and c are positive integers. The second bag contains b red sweets, c white sweets and a yellow sweets. The third bag contains c red sweets, a white sweets and b yellow sweets. Simon shakes the bags and picks one sweet from each bag without looking. Show that the probability that exactly two of the sweets are of the same colour is

$$\frac{3(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2)}{(a+b+c)^3},$$

and find the probability that the sweets are all of the same colour. Deduce that the probability that exactly two of the sweets are of the same colour is at least 6 times the probability that the sweets are all of the same colour.

13 I seat n boys and 3 girls in a line at random, so that each order of the $n+3$ children is as likely to occur as any other. Let K be the maximum number of consecutive girls in the line so, for example, $K = 1$ if there is at least one boy between each pair of girls.

(i) Find $\mathrm{P}(K = 3)$.

(ii) Show that

$$\mathrm{P}(K = 1) = \frac{n(n-1)}{(n+2)(n+3)}.$$

(iii) Find $\mathrm{E}(K)$.