Functions & Curve Sketching (STEP) (5 pages; 2/6/23)

See also:

"Cubics" (STEP)

"Transformations" (STEP)

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(B) Transformation of a simpler function

Example 1: $y = \ln(1 - x)$ is the reflection in $x = \frac{1}{2}$ of $y = lnx$

Example 2: What combination of transformations converts $y = 2^x$ to $y = 2^{4x-2}$?

Solution

$y = 2^x \rightarrow y = 2^{4x}$ is a stretch of scale factor $\frac{1}{4}$ in the x-direction

Then $y = 2^{4x} \rightarrow y = 2^{4(x-\frac{1}{2})} = 2^{4x-2}$ is a translation of $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

[Alternatively, $y = 2^{4x} \rightarrow y = \left(\frac{1}{4}\right)2^{4x} = 2^{4x-2}$ is a stretch of scale factor $\frac{1}{4}$ in the y-direction.]

(C) Symmetries of $y = f(x)$

(1) Types of symmetry

(a) about $x = a$ (special case: $x = 0$; ie y-axis)

Either $f(2a - x) = f(x)$

Or $f(a - \lambda) = f(a + \lambda)$ for all λ

[setting $x = a + \lambda$ in $f(2a - x) = f(x)$]

eg $\sin(\pi - \theta) = sin\theta$, and the sine curve has symmetry about $\theta = \frac{\pi}{2}$

[See "Transformations" (STEP)]

(b) rotational symmetry (odd function)

$f(-x) = -f(x)$

eg $\sin(-\theta) = sin\theta$

(d) symmetry about $y = x$

occurs when there is symmetry with respect to x and y;

eg $sinhx + sinhy = 1$

(2) If you are asked to sketch a curve defined for $x \in [a, b]$, consider whether it might have symmetry about the mid-point $\frac{a+b}{2}$.

(D) Greatest or least value of a function

(1) Beware of establishing the greatest or least value of a function from stationary points: these only indicate local maxima and minima.

Also, a greatest or least value may occur at a boundary of the domain.

(2) Possibilities for demonstrating that $f(x) \geq 0$

(i) $f(x) = [g(x)]^2 + [h(x)]^2$ (for all x)

(ii) For $x \geq a$: establish that $f(a) \geq 0$ and that $f'(x) \geq 0$

for $x \geq a$.

(iii) $f(x) = xsinhx[g(x)]^2$ (as x & $sinhx$ will always have the same sign - unless they are both zero) (for all x)

(E) Breaking down the domain

Example: Sketch the graph of $\sqrt{x^2 - 2x + 1}$ for $0 \leq x \leq 2$

Solution

For $0 \leq x \leq 1$, $\sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = \sqrt{(1-x)^2} = 1 - x$

For $1 \leq x \leq 2$, $\sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = x - 1$

(F) Miscellaneous

(1) For $y = |f(x)|$, when $f(x) = 0$, there will be a cusp.

Note when sketching the curve that $f'(x_0 + \delta) = -f'(x_0 - \delta)$.