STEP Exams - Methods (9 pages, 14/3/24)

Contents

(A) Generating Ideas

(1) Look ahead in the question, to get on the question-setter's wavelength.

(2) Clues in the question:

(a) If you are told that $x \neq a$, then the solution may well involve a division by $x - a$.

(b) A condition in the form of an inequality may suggest the use of $b^2 - 4ac$ (especially if it involves a squared quantity).

(c) The presence of a $\pm$ sign may suggest that a square root is being taken at some stage.

(d) There may be a clue in another part of the question.

For example, if part (i) involves 2^{2x-x^2}, the presence of $2^{-(x-c)^2}$ in part (iv) suggests that completing the square may help.

(3) Ensure that all of the information provided in the question has been used.

(4) Further Examples

(a) "Given that $c \neq 0$"

(b) "... assuming no friction" (for a car on a banked track)

(c) "... a fixed frictionless pulley" or "... in the direction of greatest slope on the plane"

(d) "Given that the magnitude of the impulsive force on the lift due to tension in the cable is equal to the magnitude of the impulsive force on the counterweight due to tension in the cable ..." [STEP 3, 2006, Q11]

There are several possibilities:

(i) The information is just to satisfy pedants, who might claim that the problem was insoluble without the given information. The examples in (c) come under this heading.

(ii) The information is there to remove a complication. Case (a) is in this category. However, we should make reference to the fact that $c \neq 0$ when, for example, dividing by c.

(iii) In (b), the absence of friction is a special case (chosen to simplify the problem).

(iv) In (d), the information is there as a hint. It can in fact be inferred from the situation in question (though there may not be time in the exam to examine it in depth).

(5) If a topic looks unfamiliar, remember that knowledge outside the syllabus is not assumed, so the question should be self-contained and include definitions of new concepts. Usually such questions turn out to be easier than normal, as the candidate is being rewarded for coping with an unfamiliar topic. Typically the first part will turn out to be quite simple.

(6) Don't do anything that is too obscure: the correct approach, once found, is usually relatively 'simple'.

(7) The first part of a STEP question may appear to be much too easy, and a trap may be suspected. But often it is just intended as a gentle introduction, to point you in the right direction. It may, for example, have been added in as an afterthought – in order not to make the question too difficult. The examiners are generally keen for students to be able to at least start a question (although they also complain when questions aren't finished).

Don't be afraid of doing something obvious. Traps aren't a feature of the STEP questions, although there may be a complication to allow for (for example, a method might not work for a particular value of a variable).

(8) The last part of a question won't necessarily be any harder than the earlier parts - especially once you have got on the question-setter's wavelength. Also, the last part might simply be the final (easy) stage in establishing an interesting result.

(B) Standard Approaches

(B.1) Creating equations

(1) Equations can be created from:

(a) information in the question

(b) relevant definitions and theorems

If necessary, create your own variables (for example, a particular length in a diagram).

Sometimes the advantage of creating an equation is that it gives you something to manipulate; ie in order to make further progress.

(2) When setting up equations or inequalities:

(i) use k^2 to represent a positive number

(ii) use $2k$ to represent an even number; $2k + 1$ to represent an odd number

(B.2) Case by case approach

(1) Example: Solve $\frac{x^2+1}{x^2-1} < 1$

Case 1: $x^2 - 1 < 0$; Case 2: $x^2 - 1 > 0$

(Once we know whether $x^2 - 1$ is positive or negative, we can multiply both sides of the inequality by it.)

(2) Transitional (or 'critical') points

This involves considering the point(s) at which the nature of a problem changes.

Example 1: $\frac{(x-1)(x+2)(x-3)}{(x+1)(x-2)(x+3)} < 0$

The only points at which the sign of the left-hand side can change are at the roots of $(x - 1)(x + 2)(x - 3) = 0$, and at the vertical asymptotes $x = -1, x = 2$ and $x = -3$

Example 2: If investigating a situation involving the intersection of a curve and a straight line, consider first the case where the line touches the curve (ie is a tangent).

(B.3) Reformulating the problem

(1) For example, in order to solve the equation $f(x) = k$, consider where the graph of $y = f(x)$ crosses the line $y = k$.

Or, more generally, rearrange an equation to $f(x) = g(x)$, and find where the curves $y = f(x)$ & $y = g(x)$ meet (or show that they won't meet).

(2) To sketch the cubic $y = x^3 + 2x^2 + x + 3$, rewrite it as

$$y = x(x^2 + 2x + 1) + 3$$

(3) Pairs of numbers x, y might be represented by the coordinates (x, y); eg to find possible values of x & y, determine the number of grid points within the relevant area.

(4) To show that a function $f(n)$ of an integer n cannot be a perfect square, show instead that $f(n) - 1$ is always a perfect square.

(5) To show that two numbers cannot be equal, show that they belong to different classes; eg even and odd numbers.

(B.4) Experimenting

(1) Often inspiration for a particular problem will only come after experimenting; or an important feature of a problem will not become apparent until then.

(2) Try things that look useful and are quick to do (ie you can quickly establish whether they are leading anywhere).

(3) Examples

(i) Draw a diagram

This may reveal a hidden feature of the problem (eg a triangle may turn out to be right-angled).

(ii) Try out particular values

Again, this may reveal a hidden feature of the problem (eg if an integer n is involved, then perhaps it has to be even).

(iii) Consider a simpler version of the problem (eg experiment with a simple function such as $y = x^2$).

(iv) List the possibilities

Often it is possible to create a list of possible cases to be considered. It may well be that the list initially seems to be prohibitively long (or even infinite), but a pattern can often emerge - or some simplying feature of the problem may become clear.

For counting problems, find a systematic way of listing the possibilities, and then of counting the items in the list.

(C) Using an earlier part of the question

(1) If one part of a question requires a result to be proved (especially if the proof is straightforward), then the answer to the next part of the question is likely to use this result.

(2) An earlier part of a question (usually the previous part) may be used in various ways:

(a) the result established previously [eg an integral] may be used in a later part of the question

(b) the method used previously [eg a substitution for an integral] may be used in a later part of the question

(c) a modification of a method used previously may be used in a later part of the question

eg a single application of Parts may be used in part (i), and a double application may be used in part (ii)

(d) An idea used previously may be used in a later part of the question

(3) A common pattern is as follows (for an integration question):

Part (i): Apply a given substitution.

Part (ii): No substitution is given, but the integral can be found by applying the same substitution.

Part (iii): No substitution is given, and the integral can be found by applying a different substitution.

(4) A rearrangement of the problem may be necessary before the result or method from a previous part can be used.

(5) When deriving a particular result R, there are two possibilities:

(a) The form of R may suggest how it is to be derived;

eg if R is $A < B$, then we might try to obtain an expression for $A - B$

(b) The information in the question may lead naturally to a result R', which can then be rearranged to give R. [See 2022, P2, Q10(ii), for example]

[Arriving at R' (rather than R) may suggest that we haven't derived the result in the way that was intended by the question setter. It may be worth stopping to see if we have missed a more direct approach. But it might be the case that R is to be used for the next part of the question, and it was intended for R' to be rearranged to produce R.]

(6) Bear in mind that, even if the question setter intended an earlier result to be used, it may not be obvious how to do this, and instead it may be easier to use a related idea instead. [See STEP

2008, P1, Q1, for example.]

(7) Further Examples:

2011, P2, Q1 (curve sketching)

2017, P1, Q1 (Integration)

2021, P3, Q3 (Integration)

2021, P3, Q12(iii) (Counting)

2022, P2, Q2 (recurrence relations)

(D) Improvements to methods

(1) A common issue with STEP questions is that a method being attempted by the candidate may not be the one envisaged by the examiners, and may be too time-consuming.

See 2021, P3, Q12(iii), for example. Having experimented with a case-by-case classification, it turns out that it is possible to count the total number of ways using a simpler formula, instead of persisting with the case-by-case approach (ie the case-by-case classification is only used to obtain an insight into the problem).

(2) If you seem to be faced with the task of almost repeating earlier work, with a slight difference in the situation, look out for a shortcut. Whereas the earlier part may have involved extensive algebra, the required proof for the new part may be very short; eg relying on a symmetry argument, or perhaps a substitution. See 2016, P1, Q9.