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Sixth Term Examination Paper [STEP]

Mathematics 3 [9475]

2020

Examiner's Report

Worked Solutions

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STEP MATHEMATICS 3

2020

Examiner's Report

STEP 3 Examiner's Report

Introduction

In spite of the change to criteria for entering the paper, there was still a very healthy number of candidates, and the vast majority handled the protocols for the online testing very well. Just over half the candidates attempted exactly six questions, and whilst about 10% attempted a seventh, hardly any did more than seven. With 20% attempting five questions, and 10% attempting only four, overall, there were very few candidates not attempting the target number. There was a spread of popularity across the questions, with no question attracting more than 90% of candidates and only one less than 10%, but every question received a good number of attempts. Likewise, there was a spread of success on the questions, though every question attracted at least one perfect solution.

Question 1

This was the most popular question, being attempted by about 90% with a fair degree of success: the mean score of about 63% made it the second best attempted question by a small margin.
In part (i), nearly all candidates understood that they would need to use integration by parts and one (or more than one, for some methods) compound angle formula. However, there were numerous manipulative errors in the integration or differentiation of the components, and even sign errors in using compound angle formulae. There were a number of different correct approaches which could be used, but they were essentially very similar to one or other of the methods in the mark scheme. Part (ii) was prescriptively worded, and it was a test of correctly expressed formalism. In spite of this, some candidates did not employ the principle of induction, some ignored that the induction was on n, and some overlooked 'non-negative' requiring the base case to be zero. Often the first component of proof by induction was omitted or incorrectly expressed. 'Assume (or suppose) the result is true for some particular k' would be an improvement on what quite a number wrote. The word 'assume (suppose)' was often not written by candidates and clearly the letter n could not be used for the assumption. Similarly, demonstrating that the base case works correctly needs to be thorough and with fully precise detail as patently it will work, and so a solution must be convincing.

Question 2

This was both the fourth most popular and successful question being attempted by 84% with a mean score of about 55%. Most generally performed much better in parts (i) and (ii) than in in (iii). In part (i), most successfully showed that were no stationary points and obtained the given result. Likewise, generally they found the points of inflection although a few struggled to do so. In part (ii), almost all candidates obtained the required equation and then noticed that it was a quadratic in e^x. Then they usually noticed that the discriminant being non-negative gave the higher bound for $\cosh a$. A surprising number seemed not to notice there was a strict lower bound to deduce, and, as a consequence, did not subsequently appreciate that a was non-zero. Given the amount of information obtained in parts (i) and (ii), there was frequently a reluctance to apply this to part (iii). For example, although stationary points usually appeared, points of inflection often did not.
Even fewer candidates used (ii) to deduce that the graph has to lie between the lines $x + y = 0$ and $x + y = \cosh^{-1}(2k^2 + 1)$. It is expected that candidates should observe that the graph is symmetrical in the line $= x$, that the two bounding lines should be labelled with their equations, and that the coordinates of the intercepts with the coordinate axes, the points of inflection and the point where it touches $x + y = \cosh^{-1}(2k^2 + 1)$ should be written in on the sketch.

Question 3

This was the second least popular pure question, but many candidates produced good solutions to it, including some very elegant ones, and the mean score was just shy of half marks. The most successful candidates were those confident in manipulating terms of the form $e^{i\theta}$. Candidates demonstrating a good knowledge of $g_{bc} - g_{ab}$ classical geometry also did well. Several candidates abandoned their attempts after part (i) or (ii).

Part (i) was generally well answered. The most common mistakes were to rotate anticlockwise rather than clockwise, or to omit the "$+a$" from their expression for k. Some candidates (including many of those with the wrong angle) still achieved the required result with no or incorrect working; as the required result was in the question, candidates could not be rewarded without correct justification.

In part (ii) most candidates attempted to show both implications at the same time, which tended to be more successful if candidates started with Q_2 being a parallelogram. Despite the question stating the order of the vertices, some candidates used an incorrect direction for one of their lines.

Most candidates used the fact that $ABCD$ is a parallelogram $\Rightarrow$ $b - a = c - d$ (or an equivalent), though a few candidates tried to use the condition that pairs of opposite sides are parallel (not appreciating the fact that if one set of sides are parallel and equal in length then the shape must be a parallelogram).

Some candidates did not appreciate that they had to state that $\omega - \omega^* \neq 0$ before being able to deduce $b - a = c - d$ from the corresponding result for Q_2. Instead, these candidates either simply cancelled $\omega - \omega^*$ without justification, or attempted (unsuccessfully) to argue that the coefficients of ω and ω^* could be directly equated.

In part (iii), some candidates stopped their attempts after finding an expression for $g_{BC} - g_{AB}$ (or similar), but most of those who attempted this part went on to produce good solutions.

There were some elegant and creative solutions to this part, but the most common approach was to try to show that $G_{CA}G_{AB}$ is a rotation of $G_{BC}G_{AB}$ by $\frac{\pi}{3}$ about G_{AB}.

A common error here was instead to try to show that $G_{CA}G_{AB}$ is a rotation of $G_{CA}G_{AB}$ by $-\frac{\pi}{3}$ about G_{AB} (sometimes arising from an incorrect labelling of T_2).

Another error which arose in several attempts was to try and compute $|g_{AB} - g_{BC}|$ by treating a, b and c as real numbers. Whilst this approach led to an expression which was totally symmetric in a, b and c, leading these candidates to 'conclude' that T_2 was equilateral, very little credit could be gained for such an approach. Only one candidate attempted to prove part (iii) independently of part (i).

Question 4

This was not a popular question but it received a respectable number of attempts with about one sixth trying it. The average score was a little under half marks, but on each part of the question, if the part was attempted it was generally fully correct. Most candidates had no problem demonstrating the desired properties, and if they used this in part (i) they had little problem obtaining full marks. Even if they could not apply the stem in (i), they nearly all found the images of **j** and **k** correctly using symmetry and hence the matrix **M** . In part (ii), almost all the candidates could solve the equations, though some lost marks by working inaccurately. The few that attempted part (iii) either got it completely correct or scored nothing: those getting it correct generally drew a parallel with the technique used in (i). As a consequence, only a small number attempted part (iv), and few scored both marks, either losing a mark for insufficient justification, or for describing the transformation as a rotation about the origin.

Question 5

A popular question, which was well attempted with a fair degree of success: it was marginally less popular than question 2, but marginally more successful. Most submitted quite a large amount of work, and were able to attempt later parts even if earlier parts were not successful as key results (requiring proof) were quoted in each part. The stem was mostly well completed, by a variety of methods, namely, re-summing indices, induction, or geometric series, though there were some candidates who seemed to think it was obvious and produced no working. Part (i) (a) was also well completed though few received full marks. The main problems were finding A and that F(x) is not defined for x = k. The second result in this part was better done, though some candidates struggled with re-summing when changing indices. For (i) (b), many did not realise that they needed to differentiate both sides. Differentiation errors and confusion thwarted many that did differentiate. Part (ii) was well done by candidates that attempted it with most realising that they could use the result of (i) (b). Though many lost marks for failing to show how to take the limit of the logarithm, most realised that they need to use partial fractions to complete the integral. Some candidates sadly left their expressions in terms of k.

Question 6

This was quite a popular question, being attempted by about three fifths of the candidates, but on average scoring only a bit better than one third marks. Most candidates were broadly successful at sketching the first graph in part (i), but though they had differentiated, many did not consider the gradients at the endpoints. Attempts to draw the sketch for part (ii) were usually less successful, and few dealt well with the behaviour near the endpoints. Few candidates gave a completely accurate justification of the small r approximation in (iii). Many candidates did not solve the equation of curve C2 for r and thus did not realise that C1 was one branch of C2. Most only drew one or other branch, and very few considered how to join the branches. Most candidates did not know how to compute areas in polar coordinates: successful ones realised the area was a difference of two polar integrals and used trigonometric substitutions to perform the integral.

Question 7

This was the second most popular question, but the most successful with a mean score of nearly two thirds marks. All but the weakest candidates managed to do part (I) perfectly well. Similarly, finding the first order differential equation for g(x) in part (ii) caused very few problems. Most candidates that attempted to substitute the given expression for g(x) in the first order differential equation obtained the correct polynomial equation, and a few gave up having done this. Most guessed the value n = -1 and then found that k = 2 works, whilst some just wrote the values of k and n, without any explanation. It wasn't uncommon for candidates to get stuck finding k or n, usually due to arithmetic errors. Most candidates attempting to find u(x) were able to find the integrating factor and perform the integration, although a significant proportion got the integral wrong. Regardless of accuracy, everyone attempted inserting the initial conditions. Some candidates also tried using a particular and complimentary solution method to integrate, but only a few who attempted that got the complimentary part correct. If candidates solved for u(x) correctly, they usually did so for y as well.

Question 8

About 60% of the candidates attempted this question, but it was the second least successful question on the paper with a mean score of about one third marks. There were some very good solutions to this question, but most candidates only provided fragmentary answers, and stopped after the first couple of parts.

A common mistake was to use the condition $u_{2k} = u_k$ along with $u_1 = 1$ to erroneously conclude that all the terms with an even subscript are equal to 1. This might have been avoided if the candidates had written out the first 10 (or so) terms of the sequence to help them get a "feel" for what was happening, which could have also help stopped some other misconceptions along the way.

Part (i) was generally well done, but some candidates did not consider both cases $u_{2k+1} > u_{2k}$ and $u_{2k-1} > u_{2k}$. Other candidates concluded that $u_{k+1} + u_k > u_k$without justifying this inequality by stating that all the terms are positive.

Part (ii) was attempted well by many candidates but was less successful than part (i). Some candidates who correctly considered both the (u_{2k-1}, u_{2k}) and (u_{2k}, u_{2k+1}) cases in part (i) then failed to consider both in this part. Some candidates erroneously assumed that if two terms p, q share a common factor and $p < q$ then it must be the case that $q = kp$.

Part (iii) was only answered well by a few of the candidates. Some did not appreciate that "consecutively" means appears one directly after another, instead taking it to mean that the second one occurs at some position after the first one. Only a small minority of attempts considered both the (u_{2k-1}, u_{2k}) and (u_{2k}, u_{2k+1}) cases. A lot of candidates erroneously stated that "if $u_k = a$ then if a is going to reappear then the next index must be $2^n k$ for some integer n". A look at the first few terms of the sequence shows that $u_5 = u_7 = 3$ which contradicts that statement.

Part (iv) was not well attempted. Some candidates did not process the wording (which was designed to help with the next part), and some tried to show instead that if *a* and *b* were two co-prime integers which **do** occur consecutively in the sequence etc.

The most successful candidates used contradiction here to show that if $a - b$ and b do occur consecutively then this means that a and b must occur consecutively.

Some candidates correctly showed the first result, but when trying to find the similar result for $a < b$ ended up with a following b and so essentially proved the same result again.

Part (v) was answered by only a few of the candidates attempting this question. There were some very well-reasoned arguments, including some candidates who used a construction method to justify that all possible rational numbers are in the range of f(*n*). Only a very small number connected part (iv) to this part of the question.

Question 9

Just over 10% of the candidates attempted this question, with most understanding the setup and writing down some resolve and moment equations. A few candidates misunderstood the setup, specifically the "(planes) meet in a horizontal line at the lowest points of both planes and lie on either side of this line" and sketched a 'tilted wedge'. There was a moderate degree of success with the mean score being just short of half marks.
Part (i) was generally done well. Candidates who were unable to progress usually forgot about the moment equation. The resolve equations were done in various ways, with the most popular being horizontal-vertical and perp-parallel to the rod. All kinds of combinations of resolve and moment equations were used. With the horizontal resolve equation and moments about the centre of mass one only needs two equations to do this part. Most candidates were able to do the required algebra, and the failure to reach the answer usually stemmed from incorrect trigonometry in the equilibrium equations.
Most candidates who succeeded in part (i) then proceeded to do part (ii). Most candidates were successful at incorporating the friction and writing down the new equations. At this point trig errors were common, and people who were resolving perp-parallel to the rod made more errors. Many candidates were put off by the difficult algebra that was about to follow. Of those who persisted, a good number arrived at the final answer, with some submitting many pages of attempts to do the algebra. The most common mistake was failing to eliminate α systematically.

Question 10

This was the least popular question on the paper being attempted by slightly less than 8% of the candidates. It was also the least successful scoring, on average, just short of one quarter marks. Four of the five results are given in the question, and many candidates tried to work backwards, albeit in disguised manners. The first results of the question related to SHM. In many cases, candidates did not clearly choose axis or positive directions, and ended with a second order differential equation without a negative sign.
It was clear that, in the next part, some did not understand that the particle, being on the platform the whole time, would have the same acceleration as the platform; when writing the equation of motion for the particle, they often included an extra force "from the platform on the particle" equal to $m\omega 2(b-x)$, using the given result. Many also just wrote down the standard equation of motion for SHM, either without having or obtaining a $b-x$ term on the RHS.
A few attempted the next section but scored no points. They understood that $R \geq 0$ for the platform to remain in contact with the particle, but at no point did they mention the range for x.
The last two sections were rarely attempted.

Question 11

Just one candidate more attempted this question than question 12, and with 20% attempting it, it was the most popular of the applied questions. Overall, there was only moderate success with the mean score just slightly better than 40%. However, there was a wide range of attempts, and although only a few obtained full marks, there were a number of strong attempts that just dropped a few marks in passing.

The first part of the question was generally well attempted, with many candidates gaining full marks. However, some struggled with the initial justification, often by failing to properly use and justify the decreasing property of the function, whilst others were led astray by attempting to find an explicit form for the function, by attempting to sketch a graph instead of providing a proof, or by failing to notice the reversal of the inequality at all.

Candidates had more difficulty with the second part of the question. Some failed to justify the use of the previous part, whilst others confused $\mathrm{f}(x)$ with the pdf of Z or Y. Many candidates correctly realised that they would need to use the strict positivity of the variance, but due to algebraic errors or other issues were unable to simplify to the required result. Finally, to receive full marks, candidates needed to ensure that relevant terms were positive in order to rearrange the inequality, which many failed to do.

Question 12

As well as the popularity of this question being similar to that of question 11, the success was very similar too. It was just below question 11 with its mean score. Very few scored full marks, partly because very few recognised the need to consider the case U=0 separately in parts (ii) and (iv), and of those who did, many made mistakes in other places or forgot to also consider it in (iv) after correctly considering it in (ii). However, the question was also rather forgiving, in the sense that it was possible to make substantial progress on the question even with errors in the earlier parts.
A common error in parts (i) and (ii) was to “double count” the case X=Y, when finding the distribution of T and U. It was also rather common for candidates to think that Y was the number of tosses until B got a tail (rather than a head). Many candidates identified correct counter-examples for the last part of (iv), but a significant proportion failed to justify that their joint probabilities were equal to zero. There were also a number of candidates who made their lives significantly harder by injudicious choice of counterexamples; e.g. candidates who chose S=2, and then U=0 who then had to do much more work to prove the probabilities were not equal, than if they had made any other choice of U would give a contradiction simply and immediately.

STEP MATHEMATICS 3

2020

Worked Solutions

STEP 3: BRIEF SOLUTIONS

1. (i) Integrating by parts,

$u = \cos^a x \qquad v' = \cos bx$
$u' = -a\cos^{a-1} x\ \sin x \qquad v = \frac{1}{b}\sin bx$

$$I(a,b) = \left[\cos^a x\ \frac{1}{b}\sin bx\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -a\cos^{a-1} x\ \sin x\ \frac{1}{b}\sin bx\ dx$$

$$= 0 + \int_0^{\frac{\pi}{2}} a\cos^{a-1} x\ \sin x\ \frac{1}{b}\sin bx\ dx = \frac{a}{b}\int_0^{\frac{\pi}{2}} \cos^{a-1} x\ \sin x\ \sin bx\ dx$$

$\cos(b-1)x = \cos bx \cos x + \sin bx \sin x$

So $I(a,b) = \frac{a}{b} \int_0^{\frac{\pi}{2}} \cos^{a-1} x\ (\cos(b-1)x - \cos bx \cos x)\ dx$
$= \frac{a}{b}[I(a-1,b-1) - I(a,b)]$

Thus $I(a,b) = \frac{a}{a+b}\ I(a-1,b-1)$ as required.

(ii) Suppose

$$I(k, k+2m+1) = (-1)^m \frac{2^k\ k!\ (2m)!\ (k+m)!}{m!\ (2k+2m+1)!}$$

Then by (i),

$$I(k+1, k+2m+2) = \frac{k+1}{2k+2m+3} I(k, k+2m+1)$$

$$= \frac{k+1}{2k+2m+3}\ (-1)^m \frac{2^k\ k!\ (2m)!\ (k+m)!}{m!\ (2k+2m+1)!}$$

$$= \frac{k+1}{2k+2m+3}\ (-1)^m \frac{2^k\ k!\ (2m)!\ (k+m)!}{m!\ (2k+2m+1)!} \times \frac{2k+2m+2}{2k+2m+2}$$

$$= (-1)^m\ \frac{2^{k+1}\ (k+1)!\ (2m)!\ (k+m+1)!}{m!\ (2k+2m+3)!}$$

$$= (-1)^m\ \frac{2^{k+1}\ (k+1)!\ (2m)!\ \big((k+1)+m\big)!}{m!\ (2(k+1)+2m+1)!}$$

which is the required result for $k+1$

$$I(0,2m+1) = \int_0^{\frac{\pi}{2}} \cos(2m+1)x \ dx = \frac{1}{2m+1} [\sin(2m+1)x]_0^{\frac{\pi}{2}}$$

$= \frac{1}{2m+1}$ if m is even or $= \frac{-1}{2m+1}$ if m is odd , or alternatively $(-1)^m \frac{1}{2m+1}$

If $n = 0$,

$$(-1)^m \frac{2^n \, n! \ (2m)! \ (n+m)!}{m! \ (2n+2m+1)!} = (-1)^m \frac{(2m)! \ (m)!}{m! \ (2m+1)!} = (-1)^m \frac{1}{2m+1}$$

so result is true for $n = 0$.

So by the principle of mathematical induction, the required result is true.

Alternative for (i)

$$\cos^a x \cos bx = \cos^{a-1} x \, [\cos x \cos bx]$$
$$\cos x \cos bx = \frac{1}{2}[\cos(b+1)x + \cos(b-1)x]$$
$$2I(a,b) = I(a-1,b+1) + I(a-1,b-1)$$

Also

$$\sin x \sin bx = \frac{1}{2}[\cos(b-1)x - \cos(b+1)x]$$

so using integration by parts of main scheme,

$$2I(a,b) = \frac{a}{b}[I(a-1,b-1) - I(a-1,b+1)]$$

Eliminating $I(a-1,b+1)$ between these results gives required result.

2. (i) $\sinh x + \sinh y = 2k$

Differentiating with respect to x, $\cosh x + \cosh y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = 0 \Rightarrow \cosh x = 0$ which is not possible as $\cosh x \geq 1 \; \forall x$, so there are no stationary points.

Differentiating again with respect to x, $\sinh x + \sinh y \left(\frac{dy}{dx}\right)^2 + \cosh y \frac{d^2y}{dx^2} = 0$

$\frac{dy}{dx} = \frac{-\cosh x}{\cosh y}$ and $\frac{d^2y}{dx^2} = 0$ implies $\sinh x + \sinh y \left(\frac{-\cosh x}{\cosh y}\right)^2 = 0$

$$\cosh^2 y \sinh x + \cosh^2 x \sinh y = 0$$

$$(1 + \sinh^2 y) \sinh x + (1 + \sinh^2 x) \sinh y = 0$$

$$(\sinh x + \sinh y)(1 + \sinh x \sinh y) = 0$$

But $\sinh x + \sinh y = 2k > 0$ so

$$1 + \sinh x \sinh y = 0$$

as required.

At a point of inflection, $\frac{d^2y}{dx^2} = 0$, so $\sinh x + \sinh y = 2k$ and $\sinh x \sinh y = -1$ and thus, $\sinh x$ (and $\sinh y$ as well) is a root of $\lambda^2 - 2k\lambda - 1 = 0$

$$\lambda = \frac{2k \pm \sqrt{4k^2 + 4}}{2}$$

$\sinh x = k + \sqrt{k^2 + 1}$, $\sinh y = \frac{-1}{k+\sqrt{k^2+1}} = \frac{-1}{k+\sqrt{k^2+1}} \times \frac{k-\sqrt{k^2+1}}{k-\sqrt{k^2+1}} = \frac{-(k-\sqrt{k^2+1})}{k^2-(k^2+1)} = k - \sqrt{k^2 + 1}$

and vice versa.

So the points of inflection are

$\left(\sinh^{-1}(k + \sqrt{k^2 + 1}),\ \sinh^{-1}(k - \sqrt{k^2 + 1})\right)$ and
$\left(\sinh^{-1}(k - \sqrt{k^2 + 1}),\ \sinh^{-1}(k + \sqrt{k^2 + 1})\right)$

(ii) $x + y = a \Rightarrow y = a - x$ so as $\sinh x + \sinh y = 2k$

$$\frac{e^x - e^{-x}}{2} + \frac{e^{a-x} - e^{x-a}}{2} = 2k$$

Multiplying by $2e^x$,

$$e^{2x} - 1 + e^a - e^{2x}e^{-a} = 4ke^x$$

$$e^{2x}(1 - e^{-a}) - 4ke^x + (e^a - 1) = 0$$

As e^x is real, $'b^2 - 4ac \geq 0'$, so $16k^2 - 4(1 - e^{-a})(e^a - 1) \geq 0$

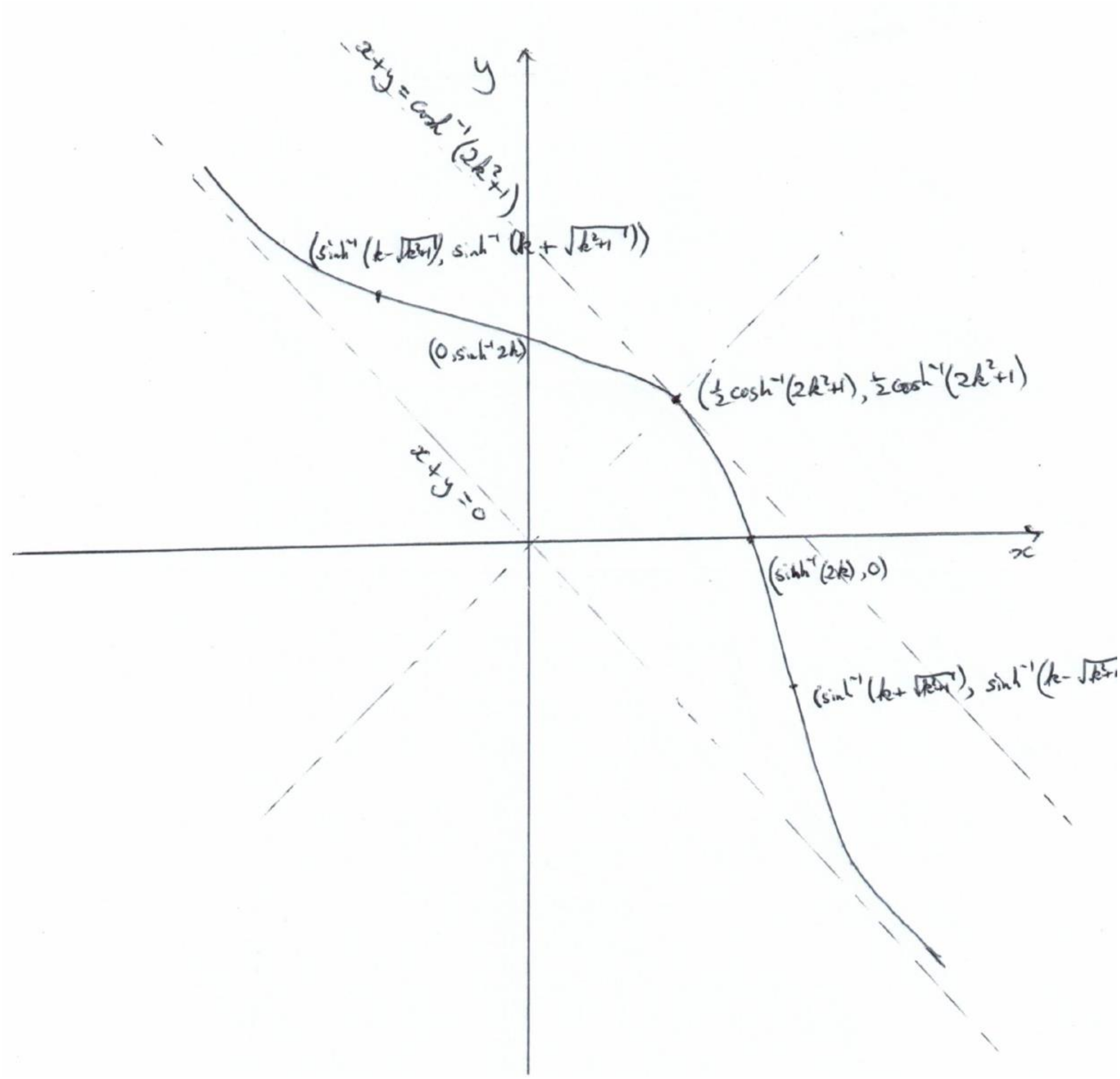
$$4k^2 - e^a - e^{-a} + 2 \geq 0$$
$$4k^2 - 2\cosh a + 2 \geq 0$$

So $\cosh a \leq 2k^2 + 1$

If $a = 0$, then $x = -y$ so $\sinh x = -\sinh y$ and thus $\sinh x + \sinh y = 2k = 0$ but $k > 0$.

So $\cosh a > 1$ as required.

(iii)



Alternative

(i) $\frac{dy}{dx} = \frac{-\cosh x}{\cosh y}$, $\frac{d^2y}{dx^2} = -\left\{\frac{\cosh y \sinh x - \cosh x \sinh y \frac{dy}{dx}}{\cosh^2 y}\right\} = -\left\{\frac{\cosh^2 y \sinh x + \cosh^2 x \sinh y}{\cosh^3 y}\right\}$

then as before.

(ii) Substituting $a = 0$ would imply $e^x = 0$ which is impossible.

3. (i)

$$k - a = (b - a)e^{-\frac{i\pi}{3}}$$

Therefore,

$$g_{AB} = \frac{1}{3}\left[a + b + \left(a + (b - a)e^{-\frac{i\pi}{3}}\right)\right]$$

$$= a\left(\frac{2 - e^{-\frac{i\pi}{3}}}{3}\right) + b\left(\frac{1 + e^{-\frac{i\pi}{3}}}{3}\right)$$

$$\omega = e^{\frac{i\pi}{6}} = \frac{\sqrt{3} + i}{2}$$

and so

$$\omega^* = \frac{\sqrt{3} - i}{2}$$

$$\frac{2 - e^{-\frac{i\pi}{3}}}{3} = \frac{2 - \left(\frac{1 - i\sqrt{3}}{2}\right)}{3} = \frac{3 + i\sqrt{3}}{6} = \frac{1}{\sqrt{3}}\frac{\sqrt{3} + i}{2} = \frac{1}{\sqrt{3}}\omega$$

and

$$\frac{1 + e^{-\frac{i\pi}{3}}}{3} = \frac{1 + \left(\frac{1 - i\sqrt{3}}{2}\right)}{3} = \frac{3 - i\sqrt{3}}{6} = \frac{1}{\sqrt{3}}\omega^*$$

Thus $g_{AB} = \frac{1}{\sqrt{3}}(\omega a + \omega^* b)$ as required.

(ii) $g_{AB} = \frac{1}{\sqrt{3}}(\omega a + \omega^* b)$
$g_{BC} = \frac{1}{\sqrt{3}}(\omega b + \omega^* c)$
$g_{CD} = \frac{1}{\sqrt{3}}(\omega c + \omega^* d)$
$g_{DA} = \frac{1}{\sqrt{3}}(\omega d + \omega^* a)$
Q_1 parallelogram $\Rightarrow b - a = c - d \Leftrightarrow d - a = c - b$

$$g_{BC} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(b - a) + \omega^*(c - b)\big) = \frac{1}{\sqrt{3}}\big(\omega(c - d\,) + \omega^*(d - a)\big) = g_{CD} - g_{DA}$$

$\Rightarrow Q_2$ parallelogram.
Q_2 parallelogram $\Rightarrow g_{BC} - g_{AB} = g_{CD} - g_{DA}$

$$\frac{1}{\sqrt{3}}\{\omega[(b - a) - (c - d\,)] + \omega^*[(c - b) - (d - a)]\} = 0$$

$$\frac{1}{\sqrt{3}}(\omega^* - \omega)[(a - b) - (d - c)] = 0$$

As $\omega^* - \omega \neq 0$, $(a - b) - (d - c) = 0$ and so Q_1 is a parallelogram

(iii)

$$g_{BC} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(b-a) + \omega^*(c-b)\big)$$

$$g_{CA} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(c-a) + \omega^*(a-b)\big) \quad (1)$$

$$\omega^2(g_{BC} - g_{AB}) = \frac{1}{\sqrt{3}}\big(\omega^3(b-a) + \omega(c-b)\big) \quad (2)$$

$$= \frac{1}{\sqrt{3}}\big(i(b-a) + \omega(c-b)\big)$$

The coefficient of $\frac{1}{\sqrt{3}}a$ in (1) is $\omega^* - \omega = \frac{\sqrt{3}-i}{2} - \frac{\sqrt{3}+i}{2} = -i$

The coefficient of $\frac{1}{\sqrt{3}}b$ in (1) is $-\omega^* = (i - \omega)$

The coefficient of $\frac{1}{\sqrt{3}}c$ in (1) is ω

Thus $G_{AB}G_{BC}$ rotated through $\frac{\pi}{3}$ is $G_{AB}G_{CA}$ which means that $G_{AB}G_{BC}G_{CA}$ is an equilateral triangle.

(iii) Alternative

$$x = g_{BC} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(b-a) + \omega^*(c-b)\big)$$

$$y = g_{CA} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(c-a) + \omega^*(a-b)\big)$$

$$x = \frac{1}{\sqrt{3}}\left(e^{\frac{i\pi}{6}}b - e^{\frac{i\pi}{6}}a + e^{\frac{-i\pi}{6}}c - e^{\frac{-i\pi}{6}}b\right)$$

$$= \frac{1}{\sqrt{3}}\left(e^{\frac{i\pi}{2}}b + e^{\frac{-i\pi}{6}}c + e^{\frac{i7\pi}{6}}a\right)$$

$$y = \frac{1}{\sqrt{3}}\left(e^{\frac{i\pi}{6}}c + e^{\frac{i3\pi}{2}}a + e^{\frac{i5\pi}{6}}b\right)$$

$$\frac{y}{x} = e^{\frac{i\pi}{3}}$$

$e^{\frac{i\pi}{3}}$ means y is x rotated through $\frac{\pi}{3}$ and thus ABC is an equilateral triangle.

[or alternatively

$\left|\frac{y}{x}\right| = 1$ and similarly $\left|\frac{z}{y}\right| = 1$ and thus all three sides are equal length]

4. π has equation $r.n = 0$ so n is a vector perpendicular to this plane.

Q lies on π if $x - (x.n)n$ satisfies $r.n = 0$

$(x - (x.n)n).n = x.n - (x.n)n.n = x.n - x.n = 0$ so Q lies on π as required.

$PQ = (x - (x.n)n) - x = -(x.n)n$ which is parallel to n and so is perpendicular to π .

(i) The image of a point with position vector x under T is $x - 2(x.n)n$, so as $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and

$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, the image of i under T is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.\begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2a\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 - 2a^2 \\ -2ab \\ -2ac \end{pmatrix}$

But $a^2 + b^2 + c^2 = 1$ so $1 - 2a^2 = a^2 + b^2 + c^2 - 2a^2 = b^2 + c^2 - a^2$

Thus, the image of i under T is $\begin{pmatrix} b^2 + c^2 - a^2 \\ -2ab \\ -2ac \end{pmatrix}$ as required.

Similarly, the images of j and k are $\begin{pmatrix} -2ab \\ c^2 + a^2 - b^2 \\ -2bc \end{pmatrix}$ and $\begin{pmatrix} -2ac \\ -2bc \\ a^2 + b^2 - c^2 \end{pmatrix}$ respectively.

Thus $M = \begin{pmatrix} b^2 + c^2 - a^2 & -2ab & -2ac \\ -2ab & c^2 + a^2 - b^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{pmatrix}$

(ii) $1 - 2a^2 = 0.\cdot 64 \Rightarrow a = \pm 0 \cdot 3\sqrt{2}$ and thus as $-2ab = 0 \cdot 48$ and $-2ac = 0 \cdot 6$,

$b = \mp 0 \cdot 4\sqrt{2}$ and $c = \mp 0 \cdot 5\sqrt{2}$ and the plane is $3x - 4y - 5z = 0$ (or $-3x + 4y + 5z = 0$)

(iii) Suppose the position vector of the point Q on the given line such that PQ is perpendicular to that line is y , then $y = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for some λ and $(y - x).\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

So, $y.\begin{pmatrix} a \\ b \\ c \end{pmatrix} - x.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$, i.e. $\lambda = x.\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

So, the image of P under the rotation, is $x + 2(y - x) = 2y - x = 2x.\begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} - x$

The image of i under the rotation is thus $\begin{pmatrix} 2a^2 - 1 \\ 2ab \\ 2ac \end{pmatrix} = \begin{pmatrix} a^2 - b^2 - c^2 \\ 2ab \\ 2ac \end{pmatrix}$, and of j and k are $\begin{pmatrix} 2ab \\ b^2 - c^2 - a^2 \\ 2bc \end{pmatrix}$ and $\begin{pmatrix} 2ac \\ 2bc \\ c^2 - a^2 - b^2 \end{pmatrix}$ respectively.

Thus $N = \begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-c^2-a^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{pmatrix}$, which, incidentally $= -M$.

(iv) $NM = -MM = -I$ as M is self-inverse.

Thus the single transformation is an enlargement, scale factor -1 , with centre of enlargement the origin.

alternative for (iii)

$x = u + v$ where $u \in \Pi$ and $v \perp \Pi$

$Mv = -v \qquad Mu = u \qquad Mx = u - v$

$Nx = v - u = -Mx$

$N = -M$

alternative for (ii) the matrix represents a reflection, an invariant point under the reflection lies on the plane of reflection.
Therefore,

$$\begin{pmatrix} 0.64 & 0.48 & 0.6 \\ 0.48 & 0.36 & -0.8 \\ 0.6 & -0.8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Taking the simplest component equation

$$0.6x - 0.8y = z$$

(although the other two give equivalent equations).

This simplifies to
$3x - 4y - 5z = 0$

5. $(x-y)(x^{n-1}+x^{n-2}y+\cdots+y^{n-1}) = x^n - x^{n-1}y + x^{n-1}y - x^{n-2}y + \cdots + xy^{n-1} - y^n$

$$= x^n - y^n$$

as each even numbered term cancels with its subsequent term.

(i) If

$$F(x) = \frac{1}{x^n(x-k)} = \frac{A}{x-k} + \frac{f(x)}{x^n}$$

then multiplying by $x^n(x-k)$

$$1 = Ax^n + (x-k)f(x)$$

$$x = k \ \Rightarrow A = \frac{1}{k^n}$$

so

$$1 = \frac{x^n}{k^n} + (x-k)f(x)$$

and

$$f(x) = \frac{1}{x-k}\left(1-\left(\frac{x}{k}\right)^n\right)$$

as required.

Thus

$$F(x) = \frac{\frac{1}{k^n}}{x-k} + \frac{\frac{1}{x-k}\left(1-\left(\frac{x}{k}\right)^n\right)}{x^n}$$

$$= \frac{1}{k^n(x-k)} - \frac{x^n-k^n}{k^nx^n(x-k)}$$

and so, by the result of the stem,

$$F(x) = \frac{1}{k^n(x-k)} - \frac{1}{k^nx^n}\sum_{r=1}^{n} x^{n-r}k^{r-1}$$

$$= \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n}\frac{1}{k^{n-r}x^r}$$

(ii)

$$x^n F(x) = \frac{1}{x-k} = \frac{x^n}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n}\frac{x^{n-r}}{k^{n-r}}$$

Differentiating with respect to x ,

$$\frac{-1}{(x-k)^2} = \frac{nx^{n-1}}{k^n(x-k)} - \frac{x^n}{k^n(x-k)^2} - \frac{1}{k}\sum_{r=1}^{n}\frac{(n-r)x^{n-r-1}}{k^{n-r}}$$

Multiplying by $\frac{-1}{x^n}$

$$\frac{1}{x^n(x-k)^2} = \frac{-n}{xk^n(x-k)} + \frac{1}{k^n(x-k)^2} + \sum_{r=1}^{n}\frac{n-r}{k^{n+1-r}x^{r+1}}$$

(iii)

$$\int_2^N \frac{1}{x^3(x-1)^2}\,dx = \int_2^N \frac{-3}{x(x-1)} + \frac{1}{(x-1)^2} + \sum_{r=1}^{3}\frac{3-r}{x^{r+1}}\ dx$$

$$= \int_2^N \frac{3}{x} - \frac{3}{(x-1)} + \frac{1}{(x-1)^2} + \sum_{r=1}^{3}\frac{3-r}{x^{r+1}}\ dx$$

$$= \left[3\ln x - 3\ln(x-1) - \frac{1}{x-1} - \sum_{r=1}^{3}\frac{3-r}{rx^r}\right]_2^N$$

$$= \left[3\ln\left(\frac{x}{x-1}\right) - \frac{1}{x-1} - \sum_{r=1}^{3}\frac{3-r}{rx^r}\right]_2^N$$

$$= 3\ln\left(\frac{N}{N-1}\right) - \frac{1}{N-1} - \sum_{r=1}^{3}\frac{3-r}{rN^r} - 3\ln(2) + 1 + \sum_{r=1}^{3}\frac{3-r}{r2^r}$$

As $N \to \infty$, $\left(\frac{N-1}{N}\right) \to 1$, so $3\ln\left(\frac{N-1}{N}\right) \to 0$, and $\frac{1}{N-1} \to 0$, $\frac{1}{N^r} \to 0$

So the limit of the integral is,

$$-3\ln 2 + 1 + \frac{2}{2} + \frac{1}{8} = -3\ln 2 + \frac{17}{8}$$

Alternatives for stem using sum of GP or proof by induction

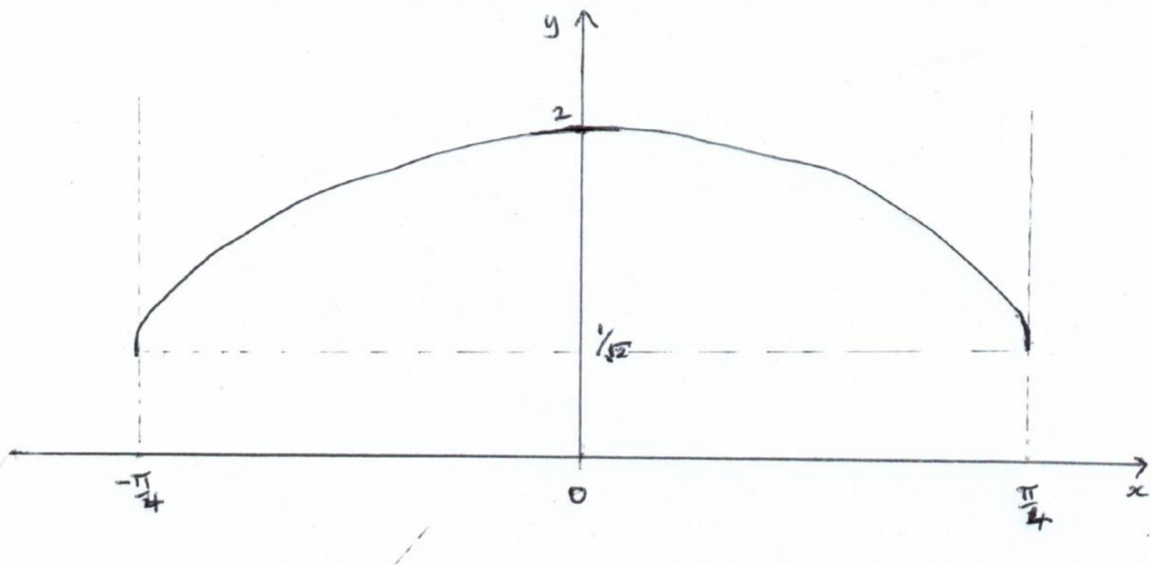
6. (i)

$$y = \cos x + \sqrt{\cos 2x}$$

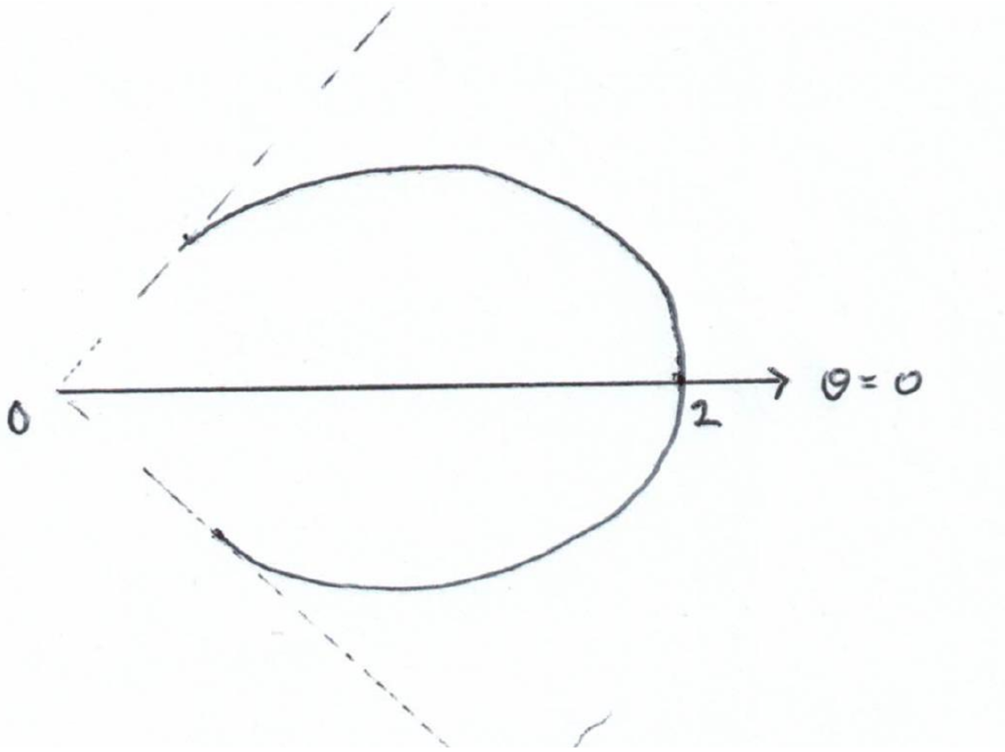
$x = 0\,, y = 2$ There is symmetry in $x = 0$ $x = \pm\frac{\pi}{4}\,, y = \frac{1}{\sqrt{2}}$

$\frac{dy}{dx} = -\sin x - \frac{\sin 2x}{\sqrt{\cos 2x}}$ so $x = 0\,, \frac{dy}{dx} = 0$ $x > 0\,, \frac{dy}{dx} < 0$ and vice versa

as $x \to \frac{\pi}{4}\,, \frac{dy}{dx} \to -\infty$



(ii)



(iii) $\theta = \pm\frac{\pi}{4}$, $r = \frac{1}{\sqrt{2}}$

$r^2 - 2r\cos\theta + \sin^2\theta = 0$

$$(r - \cos\theta)^2 = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

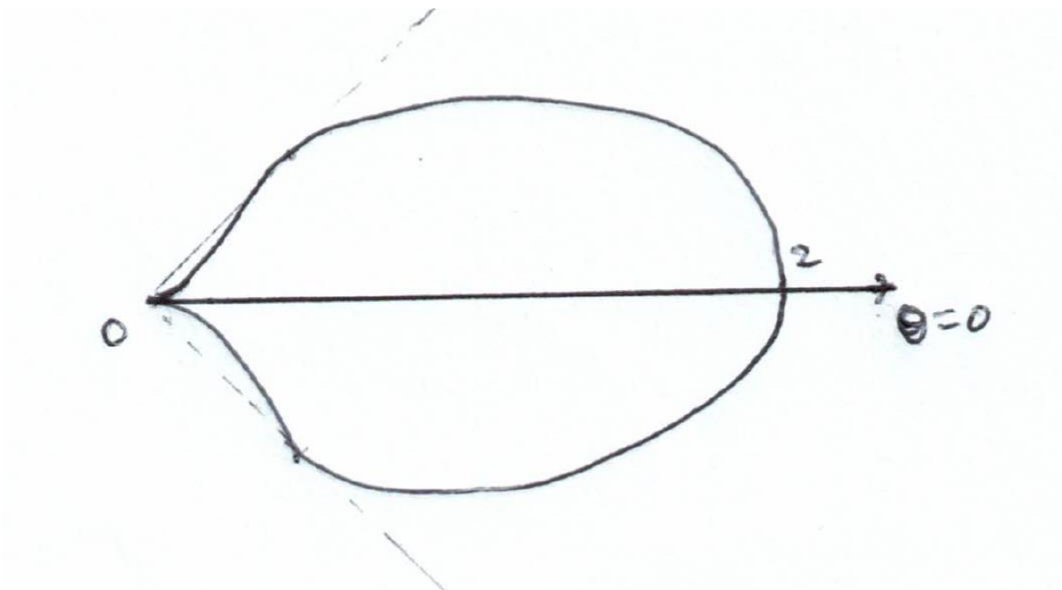
Therefore, $r - \cos\theta = \pm\sqrt{\cos 2\theta}$, i.e. $r = \cos\theta \pm \sqrt{\cos 2\theta}$

From (i), r is only small on the branch, $r = \cos\theta - \sqrt{\cos 2\theta}$. For $\theta = 0$, $r = 0$

Otherwise, $\cos\theta - \sqrt{\cos 2\theta} = 0$, $\cos 2\theta = \cos^2\theta$, $2\cos^2\theta - 1 = \cos^2\theta$, $\cos\theta = \pm 1$ so for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, $\theta = 0$ is the only value for which $r = 0$

So r small implies $r = \cos\theta - \sqrt{\cos 2\theta}$ and θ is small

Thus $r \approx 1 - \frac{\theta^2}{2} - \left(1 - \frac{(2\theta)^2}{2}\right)^{\frac{1}{2}} \approx 1 - \frac{\theta^2}{2} - 1 + \theta^2 = \frac{\theta^2}{2}$ as required.



Area required is

$$\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\cos\theta + \sqrt{\cos 2\theta}\right)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\cos\theta - \sqrt{\cos 2\theta}\right)^2 d\theta$$

$$= 2\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{\cos 2\theta}\ d\theta$$

$$= 2\int_0^{\frac{\pi}{4}}\cos\theta\ \sqrt{1 - 2\sin^2\theta}\ d\theta$$

Let $\sqrt{2}\ \sin\theta = \sin u$, then $\sqrt{2}\ \cos\theta\ \frac{d\theta}{du} = \cos u$, So the integral becomes

$$2\int_0^{\frac{\pi}{2}} \frac{\cos^2 u}{\sqrt{2}}\ du = \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{\cos 2u + 1}{2}\ du = \sqrt{2}\left[\frac{\sin 2u}{4} + \frac{u}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2\sqrt{2}}$$

(iii) alternative

$$r \ll 1 \Rightarrow -2r\cos\theta + \sin^2\theta \approx 0$$
$$r \approx \frac{\sin^2\theta}{2\cos\theta} = \frac{1}{2}\sin\theta\tan\theta \approx \frac{\theta^2}{2}$$

7. (i) $u = \frac{dy}{dx} + g(x)y$

Thus

$$\frac{du}{dx} = \frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + g'(x)y$$

As $\frac{du}{dx} + f(x)u = h(x)$

$$\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + g'(x)y + f(x)\left(\frac{dy}{dx} + g(x)y\right) = h(x)$$

that is

$$\frac{d^2y}{dx^2} + \big(g(x) + f(x)\big)\frac{dy}{dx} + \big(g'(x) + f(x)g(x)\big)y = h(x)$$

as required.

(ii)

$$g(x) + f(x) = 1 + \frac{4}{x}$$

and so $f(x) = 1 + \frac{4}{x} - g(x)$

$$g'(x) + f(x)g(x) = \frac{2}{x} + \frac{2}{x^2}$$

so

$$g'(x) + \left(1 + \frac{4}{x} - g(x)\right)g(x) = \frac{2}{x} + \frac{2}{x^2}$$

as requested.

If $g(x) = kx^n$, $g'(x) = knx^{n-1}$

$$knx^{n-1} + \left(1 + \frac{4}{x} - kx^n\right)kx^n = \frac{2}{x} + \frac{2}{x^2}$$

$$-k^2x^{2n+2} + kx^{n+2} + k(n+4)x^{n+1} - 2x - 2 = 0$$

Considering the x^{2n+2} term,

either it is eliminated by the x^{n+2} term, in which case, $2n + 2 = n + 2$ and $-k^2 + k = 0$

which would imply $n = 0$ and $k = 0$ or $k = 1$

$k = 0$ is not possible $(-2x - 2 = 0)$; $n = 0$, $k = 1$ would give $4x - 2x - 2 = 0$ so not possible

Or it is eliminated by the x^{n+1} term, in which case, $2n+2=n+1$ which implies $n=-1$ and thus $-k^2+k(n+4)-2=0$ and considering the other two terms $k-2=0$

$k=2$ and $n=-1$ satisfy $-k^2+k(n+4)-2=0$ so these are possible values.

So $g(x)=\frac{2}{x}$ and as $f(x)=1+\frac{4}{x}-g(x)$, $f(x)=1+\frac{2}{x}$ $h(x)=4x+12$

$\frac{du}{dx}+f(x)u=h(x)$ is thus $\frac{du}{dx}+\left(1+\frac{2}{x}\right)u=4x+12$

The integrating factor is

$$e^{\int\left(1+\frac{2}{x}\right)dx}=e^{x+2\ln x}=x^2e^x$$

Thus

$$x^2e^x\frac{du}{dx}+(x^2+2x)e^xu=(4x+12)x^2e^x=(4x^3+12x^2)e^x$$

Integrating with respect to x

$$x^2e^xu=\int(4x^3+12x^2)e^x\ dx=4x^3e^x+c$$

As $u=\frac{dy}{dx}+g(x)y$, and $g(x)=\frac{2}{x}$, when $x=1$, $y=5$, $\frac{dy}{dx}=-3$, we have $u=-3+2\times 5$

That is $u=7$, so $7e=4e+c$, which means $c=3e$

So $u=4x+3e\ \frac{e^{-x}}{x^2}$

$$\frac{dy}{dx}+\frac{2}{x}y=4x+3e\ \frac{e^{-x}}{x^2}$$

This has integrating factor

$$e^{\int\frac{2}{x}dx}=e^{2\ln x}=x^2$$

So

$$x^2\frac{dy}{dx}+2xy=4x^3+3e\ e^{-x}$$

Integrating with respect to x

$$x^2y=\int 4x^3+3e\ e^{-x}\ dx=\ x^4-3ee^{-x}+c'$$

when $x=1$, $y=5$ so $5=1-3+c'$ which means $c'=7$

Therefore,
$y=x^2+\frac{7}{x^2}-\frac{3e^{-x+1}}{x^2}$

8. (i) All terms of the sequence are positive integers because they are all either equal to a previous term or the sum of two previous terms which are positive integers.

Thus, for $k \geq 1$, as $u_{2k} = u_k$ and $u_{2k+1} = u_k + u_{k+1}$, $u_{2k+1} - u_{2k} = u_{k+1} \geq 1$

Also, $u_{2k+1} - u_{2k+2} = u_k + u_{k+1} - u_{k+1} = u_k \geq 1$. Thus, the required result is proved for terms from the third onwards. (The only terms not included in this proof are the first two, which are in case both equal to 1).

(ii) Suppose that $u_{2k} = c$, and that $u_{2k+1} = d$, for $k \geq 1$, where d and c share a common factor greater than one, then $u_k = c$, as $u_{2k} = u_k$, and $u_{k+1} = d - c \geq 1$ as $u_{2k+1} = u_k + u_{k+1}$ and using (i). Then as d and c share a common factor greater than one, d-c and c share a common factor greater than one. So, two earlier terms in the sequence do share the same common factor.

Likewise, suppose that $u_{2k+2} = c$, and that $u_{2k+1} = d$, for $k \geq 1$, where d and c share a common factor greater than one, then $u_{k+1} = c$ and $u_k = d - c$ giving the same result.

This is true for pairs of consecutive terms from the second term (and third) onwards. Repeating this argument, we find that it would imply that the first two terms would share a common factor greater than one, which is a contradiction. Hence any two consecutive terms are co-prime.

(iii) For $k \geq 1$, and $m \geq 1$ suppose that $u_{2k} = c$ and $u_{2k+1} = d$, and that $u_{2k+m} = c$ and $u_{2k+m+1} = d$, then as $d > c$, $2k + m$ is even, so m is even, say $2n$. Thus, $u_k = c$ and $u_{k+1} = d - c$, and $u_{k+n} = c$ and $u_{k+n+1} = d - c$. That is, an earlier pair of terms would appear consecutively.

Likewise, if $u_{2k+2} = c$ and $u_{2k+1} = d$, and that $u_{2k+m+2} = c$ and $u_{2k+m+1} = d$, the same argument applies.

So the argument can be repeated down to the first two terms, which are of course equal, and it would imply a later pair are likewise which contradicts (i).

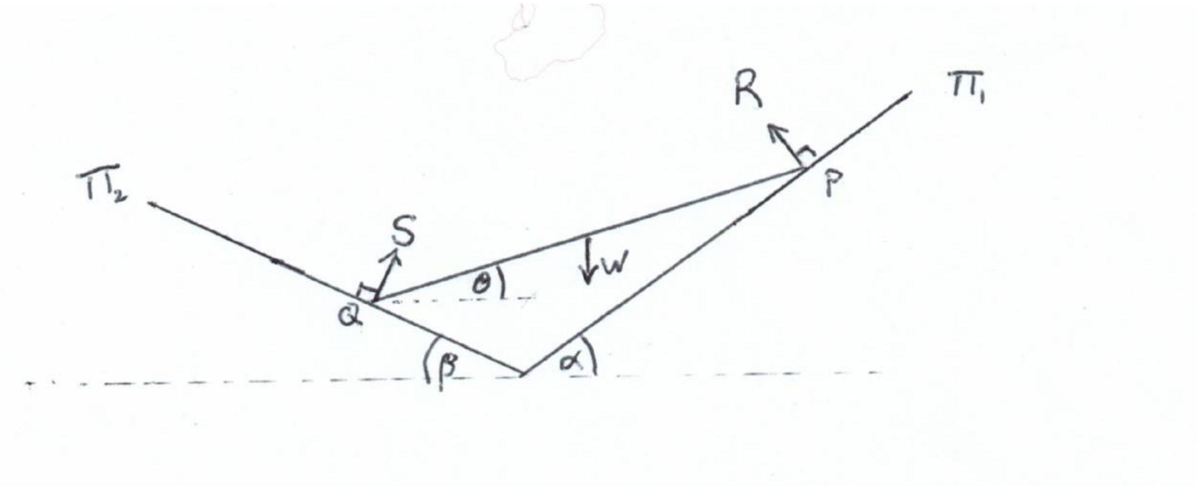
(iv) If (a, b) does not occur, where a and b are coprime and $a > b$, then there does not exist k such that $u_{2k+1} = a$ and $u_{2k+2} = b$. Therefore there cannot exist a *k* such that $u_{k+1} = b$ and $u_k = a - b$, the sum of which is a , which is smaller than $a + b$.

If (a, b) does not occur, where a and b are coprime and $a < b$, then there does not exist k such that $u_{2k} = a$ and $u_{2k+1} = b$. Therefore there cannot exist a *k* such that $u_k = a$ and $u_{k+1} = b - a$, the sum of which is b, which is smaller than $a + b$.

(v) Suppose that there exists an ordered pair of coprime integers (a,b) which does not occur consecutively in the sequence. Then by part (iv) the pair (a-b, b) [if a>b] or (a, b-a) [if b>a] (which has a smaller sum) does not occur. Repeating this means that a coprime pair with sum <3 does not occur. The only coprime pair of integers with sum <3 is (1, 1) which are the first two terms. Contradiction and so every ordered pair of coprime integers occurs in the sequence and by (iii) only occurs once. Therefore, there exists an n , and that n is unique such that

$q = \frac{u_n}{u_{n+1}}$, for any positive rational q (which is expressed in lowest form). So the inverse of f exists.

9. (i)



Resolving vertically, (1)

$$R\cos\alpha + S\cos\beta = W$$

Resolving horizontally, (2)

$$R\sin\alpha = S\sin\beta$$

Taking moments about Q, (3)

$$Wl\cos\theta = 2Rl\cos(\alpha - \theta)$$

Dividing (3) by l gives (4)

$$W\cos\theta = 2R\cos(\alpha - \theta)$$

Multiplying (1) by $\cos\theta\ \sin\beta$ gives

$$R\cos\alpha\ \cos\theta\ \sin\beta + S\cos\beta\cos\theta\ \sin\beta = W\cos\theta\ \sin\beta$$

Using (2) and (4) to substitute for $S\sin\beta$ and $W\cos\theta$ respectively,

$$R\cos\alpha\ \cos\theta\ \sin\beta + R\sin\alpha\cos\beta\cos\theta = 2R\cos(\alpha - \theta)\ \sin\beta$$

Thus

$$\cos\alpha\ \cos\theta\ \sin\beta + \sin\alpha\cos\beta\cos\theta = 2\cos\alpha\ \cos\theta\ \sin\beta + 2\sin\alpha\ \sin\theta\sin\beta$$

and so

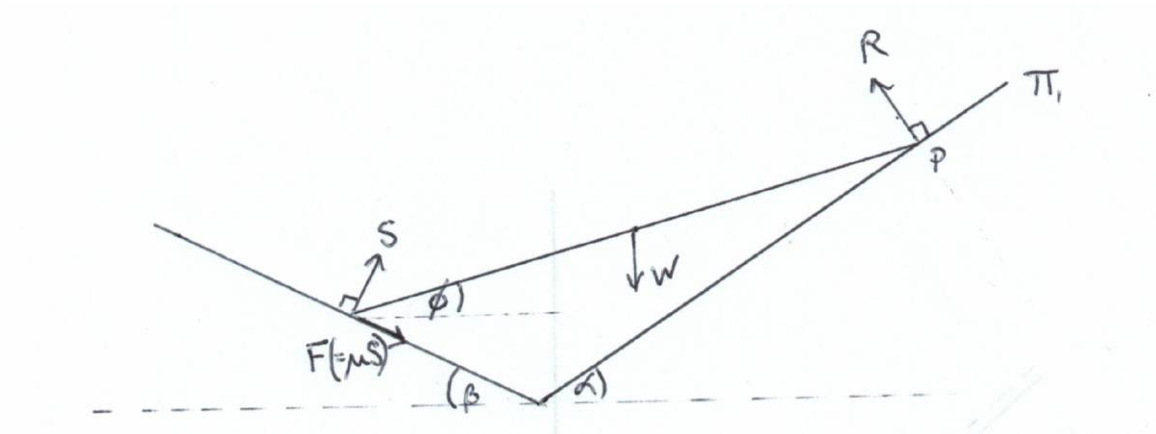
$$\sin\alpha\cos\beta\cos\theta - \cos\alpha\ \cos\theta\ \sin\beta = 2\sin\alpha\ \sin\theta\sin\beta$$

Dividing by $\sin\alpha\ \cos\theta\sin\beta$ gives

$$\cot\beta - \cot\alpha = 2\tan\theta$$

as required.

(ii)



Resolving vertically, (5)

$$R\cos\alpha + S\cos\beta = W + \mu S\sin\beta$$

Resolving horizontally, (6)

$$R\sin\alpha = S\sin\beta + \mu S\cos\beta$$

Taking moments about Q, and dividing by by l gives as before (4)

$$W\cos\phi = 2R\cos(\alpha-\phi)$$

Multiplying (5) by $(\sin\beta + \mu\cos\beta)\cos\phi$ gives

$$\begin{aligned}R\cos\alpha\,(\sin\beta+\mu\cos\beta)\cos\phi + S\cos\beta\,(\sin\beta+\mu\cos\beta)\cos\phi\\ = W(\sin\beta+\mu\cos\beta)\cos\phi + \mu S\sin\beta\,(\sin\beta+\mu\cos\beta)\cos\phi\end{aligned}$$

Using (6) and (4) to substitute for $S(\sin\beta+\mu\cos\beta)$ and $W\cos\phi$ respectively,

$$\begin{aligned}R\cos\alpha\,(\sin\beta+\mu\cos\beta)\cos\phi + R\sin\alpha\cos\beta\cos\phi\\ = 2R\cos(\alpha-\phi)(\sin\beta+\mu\cos\beta) + \mu\,R\sin\alpha\sin\beta\cos\phi\end{aligned}$$

Thus

$$\begin{aligned}\cos\alpha\,(\sin\beta+\mu\cos\beta)\cos\phi + \sin\alpha\cos\beta\cos\phi\\ = 2(\cos\alpha\cos\phi + \sin\alpha\ \sin\phi)(\sin\beta+\mu\cos\beta) + \mu\ \sin\alpha\sin\beta\cos\phi\end{aligned}$$

So, dividing by $\sin\alpha\cos\beta\ \cos\phi$,

$$\cot\alpha\,(\tan\beta+\mu) + 1 = 2\cot\alpha\,(\tan\beta+\mu) + 2\tan\phi\ (\tan\beta+\mu) + \mu\tan\beta$$

$$1 = \cot\alpha\,(\tan\beta+\mu) + 2\tan\phi\ (\tan\beta+\mu) + \mu\tan\beta$$

From (i), $\cot\alpha = \cot\beta - 2\tan\theta$

so

$$1 = (\cot\beta - 2\tan\theta)(\tan\beta+\mu) + 2\tan\phi\ (\tan\beta+\mu) + \mu\tan\beta$$

$$1 - \mu\tan\beta - 1 - \mu\cot\beta = 2(\tan\phi - \tan\theta)(\tan\beta+\mu)$$

Hence,

$$2(\tan\theta - \tan\phi)(\tan\beta+\mu) = \mu\tan\beta + \mu\cot\beta = \mu(\tan\beta+\cot\beta)$$

$$\tan\beta + \cot\beta = \frac{\sin\beta}{\cos\beta} + \frac{\cos\beta}{\sin\beta} = \frac{\sin^2\beta + \cos^2\beta}{\sin\beta\ \cos\beta} = \frac{1}{\sin\beta\ \cos\beta}$$

and so,

$$\tan\theta - \tan\phi = \frac{\mu}{2\sin\beta\ \cos\beta(\tan\beta + \mu)}$$

That is

$$\tan\theta - \tan\phi = \frac{\mu}{(\mu + \tan\beta)\sin 2\beta}$$

as required.

Alternative method using concurrency principle

10. If the extension in the equilibrium position is , then

$$mg = \frac{kmgd}{a}$$

Thus, $d = \frac{a}{k}$

If the extension when the particle is released is $d + x$, then the equation of motion is

$$m\ddot{x} = mg - \frac{kmg(d+x)}{a} = mg - \frac{kmgd}{a} - \frac{kmgx}{a} = -\frac{kmgx}{a}$$

$$\ddot{x} = -\frac{kgx}{a}$$

This is simple harmonic motion with period $\frac{2\pi}{\Omega}$ where $\Omega^2 = \frac{kg}{a}$, i.e. $kg = a\Omega^2$ as required.

Let y be the displacement of the platform below the centre point of its oscillation,

then, $y = b - x$ and $\ddot{y} = -\omega^2 y = -\omega^2(b - x)$ (x newly defined as in question)

Thus, the equation of motion of the particle becomes

$$m\,\ddot{y} = mg - R - \frac{kmg(h - a - x)}{a}$$

So,

$$-m\omega^2(b - x) = mg - R - \frac{ma\Omega^2(h - a - x)}{a}$$

That is,

$$R = mg + m\Omega^2(a + x - h) + m\omega^2(b - x)$$

as required.

To remain in contact, $R \geq 0$ for $0 \leq x \leq 2b$

$$R = mg + m\Omega^2(a - h) + m\omega^2 b + m(\Omega^2 - \omega^2)x$$

so if $\omega < \Omega$, the minimum value of R is $mg + m\Omega^2(a - h) + m\omega^2 b$, (when $x = 0$)

thus $mg + m\Omega^2(a - h) + m\omega^2 b \geq 0$

Rearranging,

$$h \leq \frac{g + \omega^2 b}{\Omega^2} + a = \frac{a}{k} + \frac{\omega^2 b}{\Omega^2} + a = a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}$$

as required.

If $\omega > \Omega$, minimum value of R is $mg + m\Omega^2(a - h) + m\omega^2 b + 2mb(\Omega^2 - \omega^2)$ (at $x = 2b$)

Thus,

$$mg + m\Omega^2(a-h) + m\omega^2 b + 2mb(\Omega^2 - \omega^2) \geq 0$$

and so,

$$h \leq \frac{g+\omega^2 b}{\Omega^2} + a + \frac{2b(\Omega^2-\omega^2)}{\Omega^2} = a\left(1+\frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2} + 2b - \frac{2\omega^2 b}{\Omega^2} = a\left(1+\frac{1}{k}\right) - \frac{\omega^2 b}{\Omega^2} + 2b$$

Thus, if $\omega < \Omega$, $h \leq a\left(1+\frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2} < a\left(1+\frac{1}{k}\right) + b$;

if $\omega > \Omega$, $h \leq a\left(1+\frac{1}{k}\right) - \frac{\omega^2 b}{\Omega^2} + 2b < a\left(1+\frac{1}{k}\right) + b$

If $\omega = \Omega$, then $R = mg + m\Omega^2(a-h) + m\omega^2 b \geq 0$,

so,

$$h \leq a\left(1+\frac{1}{k}\right) + b$$

Alternative

stem measuring y below A

$$m\ddot{y} = mg - \frac{kmg(y-a)}{a}$$

$$\ddot{y} = -\frac{kg}{a}\, y + kg\left(1+\frac{1}{k}\right)$$

and for platform introduced

$$m\,\ddot{y} = mg - R - \frac{kmg(y-a)}{a}$$

11. (i)

$$P(Y \le y) = P(f(X) \le y)$$
$$= P\left(X \ge f^{-1}(y)\right)$$

as f is a strictly decreasing function

$$= P\left(X \ge f(y)\right)$$

$$= \frac{b - f(y)}{b - a}$$

because X is uniformly distributed on [a,b].

Thus, the pdf of Y is

$$\frac{d}{dy}\left(\frac{b - f(y)}{b - a}\right) = \frac{-f'(y)}{b - a}$$

$$y \in [a, b]$$

$$E(Y^2) = \int_a^b y^2 \frac{-f'(y)}{b - a}\, dy = \left[y^2 \frac{-f(y)}{b - a}\right]_a^b - \int_a^b 2y \frac{-f(y)}{b - a}\, dy$$

by integration by parts

$$= \frac{-ab^2 + a^2 b}{b - a} + \int_a^b 2x \frac{f(x)}{b - a}\, dx = \frac{ab(a - b)}{b - a} + \int_a^b 2x \frac{f(x)}{b - a}\, dx = -ab + \int_a^b 2x \frac{f(x)}{b - a}\, dx$$

as required.

(ii) Considering Z as a function of X, it satisfies the three conditions for the function f in part (i), as trivially by the definition of c the first is satisfied, considering the graph or the derivative the second is, and by symmetry, the third is.

$$\frac{1}{Z} + \frac{1}{X} = \frac{1}{c}$$

so

$$\frac{1}{Z} = \frac{1}{c} - \frac{1}{X} = \frac{X - c}{cX}$$

and therefore

$$Z = \frac{cX}{X - c}$$

Therefore,

$$E(Z) = \int_a^b \frac{cx}{x - c} \frac{1}{b - a}\, dx = \frac{c}{b - a} \int_a^b \frac{x - c}{x - c} + \frac{c}{x - c}\, dx = \frac{c}{b - a} \int_a^b 1 + \frac{c}{x - c}\, dx$$

$$= \frac{c}{b-a}\ [x + \mathrm{c}\ln(x-c)]_a^b$$

$$= c + \frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)$$

From (i),

$$E(Z^2) = -ab + \int_a^b \frac{cx}{x-c}\,\frac{2x}{b-a}\ dx$$

$$= -ab + \frac{2c}{b-a}\int_a^b \frac{x^2}{x-c}\ dx = -ab + \frac{2c}{b-a}\int_a^b \frac{x^2-xc}{x-c} + \frac{xc-c^2}{x-c} + \frac{c^2}{x-c}\ dx$$

$$= -ab + \frac{2c}{b-a}\left[\frac{x^2}{2} + cx + c^2\ln(x-c)\right]_a^b$$

$$= -ab + c(a+b) + 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right)$$

$$= -ab + ab + 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right) = 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right)$$

Thus,

$$Var(Z) = 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right) - \left(c + \frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)\right)^2$$

$$= c^2 - \left(\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)\right)^2$$

As $Var(Z) > 0$, (Z is not a constant)

$$c^2 - \left(\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)\right)^2 > 0$$

and so as c and $\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)$ are both positive,

$$c > \frac{c^2}{b-a} \ln\left(\frac{b-c}{a-c}\right)$$

and similarly,

$$\ln\left(\frac{b-c}{a-c}\right) < \frac{b-a}{c}$$

as required.

12. (i) $P(X=x)=q^{x-1}p$ and $P(Y=y)=q^{y-1}p$ for $x,y\geq 1$

$$P(S=s)=P(X+Y=s)=\sum_{x=1}^{s-1}P(X=x,Y=s-x)=\sum_{x=1}^{s-1}q^{x-1}pq^{s-x-1}p$$

$$=\sum_{x=1}^{s-1}q^{s-2}p^2=(s-1)p^2q^{s-2}$$

for $s\geq 2$.

$$P(T=t)=P(X=t,Y\leq t)+P(Y=t,X\leq t)-P(X=t,Y=t)$$

$$=2q^{t-1}p\sum_{y=1}^{t}q^{y-1}p\;-q^{t-1}pq^{t-1}p$$

$$=2q^{t-1}p^2\,\frac{1-q^t}{1-q}-q^{2t-2}\,p^2=2q^{t-1}p(1-q^t)-q^{2t-2}\,p^2$$

$$=q^{t-1}p\,(2-2q^t-q^{t-1}p)=pq^{t-1}(2-2q^t-(1-q)q^{t-1})=\;pq^{t-1}(2-q^{t-1}-q^t)$$

for $t\geq 1$ as required.

(ii)

$$P(U=u)=\sum_{x=1}^{\infty}P(X=x,Y=x+u)+\sum_{x=1}^{\infty}P(Y=x,X=x+u)$$

for $u\geq 1$

$$=2\sum_{x=1}^{\infty}q^{x-1}p\,q^{x+u-1}p=\;2p^2\,q^u\sum_{x=1}^{\infty}q^{2x-2}=2p^2\,q^u\,\frac{1}{1-q^2}$$

$$=2p^2\,q^u\,\frac{1}{p(1+q)}=\frac{2p\,q^u}{(1+q)}$$

and

$$P(U=0)=\sum_{x=1}^{\infty}P(X=x,Y=x)=\;\sum_{x=1}^{\infty}q^{x-1}p\,q^{x-1}p=\;p^2\sum_{x=1}^{\infty}q^{2x-2}=\frac{p^2}{1-q^2}=\frac{p}{1+q}$$

$$P(W = w) = P(X = w, Y \geq w) + P(Y = w, X \geq w) - P(X = w, Y = w)$$

for $w \geq 1$

$$= 2\sum_{y=0}^{\infty} q^{w-1}p\, q^{w+y-1}p - q^{w-1}p\, q^{w-1}p = 2p^2q^{2w-2}\sum_{y-0}^{\infty} q^y - p^2q^{2w-2}$$

$$= p^2q^{2w-2}\, \frac{2}{1-q} - p^2q^{2w-2} = p^2q^{2w-2}\left(\frac{2}{1-q} - 1\right) = p^2q^{2w-2}\frac{1+q}{1-q}$$

$$= pq^{2w-2}(1+q)$$

(iii) $S = 2 \Rightarrow X = 1, Y = 1$ and $T = 3 \Rightarrow X = 3$ or $Y = 3$ or both

Thus,

$P(S = 2, T = 3) = 0$

However,

$P(S = 2) = p^2 \neq 0$ and $P(T = 3) = pq^2(2 - q^2 - q^3) = pq^2(1 - q^2 + 1 - q^3)$

$$= pq^2\big(p(1+q) + p(1+q+q^2)\big)$$

$$= p^2q^2(2 + 2q + q^2) \neq 0$$

and so, $P(S = 2, T = 3) \neq P(S = 2) \times P(T = 3)$

(iv)

$U = u$ and $W = w \Rightarrow X = w, Y = w + u$ or $Y = w, X = w + u$ for $u > 0$

So $P(U = u, W = w) = 2q^{w-1}pq^{w+u-1}p = 2p^2q^{2w+u-2}$

$$P(U = u) = \frac{2p\, q^u}{(1+q)}$$

and

$$P(W = w) = pq^{2w-2}(1+q)$$

so

$$P(U = u) \times P(W = w) = \frac{2p\, q^u}{(1+q)} \times pq^{2w-2}(1+q) = 2p^2q^{2w+u-2} = P(U = u, W = w)$$

In the case $u = 0$,

$U = 0$ and $W = w \Rightarrow X = w, Y = w$

so $P(U = 0, W = w) = q^{w-1}p \quad q^{w-1}p \ = p^2q^{2w-2}$

whereas $P(U = 0) = \frac{p}{1+q}$ and $P(W = w) = pq^{2w-2}(1+q)$

so

$$P(U=0)\times P(W=w)=\frac{p}{1+q}\times pq^{2w-2}(1+q)=p^2q^{2w-2}=P(U=0\,,W=w)$$

Thus, U and W are independent.

Pairing S and U – consider $S=2$, $U=3$

$S=2\ \Rightarrow X=1\,,Y=1$ which would imply $U=0$

Thus $P(S=2)=p^2\neq 0$ and $P(U=3)=\frac{2p\,q^3}{(1+q)}\neq 0$ whereas $P(S=2\,,U=3)=0$ so

$P(S=2\,,U=3)\ \neq\ P(S=2)\ \times\ P(U=3)$ and S and U are not independent.

Pairing S and W - consider $S=2$, $W=3$

$S=2\ \Rightarrow X=1\,,Y=1$ which would imply $W=1$

so $P(S=2\,,W=3)=0$ whereas $P(S=2)\neq 0$ and $P(W=3)\ \neq 0$ so S and W are not independent.

Pairing T and U - consider $T=1$, $U=1$

$T=1\ \Rightarrow X=1\,,Y=1$ which would imply $U=0$

so $P(T=1\,,U=1)=0$ whereas $P(T=1)\neq 0$ and $P(U=1)\ \neq 0$ so T and U are not independent.

Pairing T and W - consider $T=1$, $W=2$

$T=1\ \Rightarrow X=1\,,Y=1$ which would imply $W=1$

so $P(T=1\,,W=2)=0$ whereas $P(T=1)\neq 0$ and $P(W=2)\ \neq 0$ so T and W are not independent.

Alternative (i)

$$P(T=t)=P(X=t,Y<t)+P(Y=t,X<t)+P(X=t,Y=t)$$

[or $P(T=t)=P(X=t,Y<t)+P(Y=t,X\leq t)$]

$$=2q^{t-1}p\sum_{y=1}^{t-1}q^{y-1}p\ +q^{t-1}pq^{t-1}p$$

$$=2q^{t-1}p^2\,\frac{1-q^{t-1}}{1-q}+q^{2t-2}\,p^2=2q^{t-1}p(1-q^{t-1})+q^{2t-2}\,p^2$$

$$= q^{t-1}p\,(2 - 2q^{t-1} + q^{t-1}p) = pq^{t-1}(2 - 2q^{t-1} + (1-q)q^{t-1}) = \; pq^{t-1}(2 - q^{t-1} - q^{t})$$

for $t \geq 1$ as required.

(ii)

$$P(W = w) = P(X = w, Y > w) + P(Y = w, X > w) + P(X = w, Y = w)$$

for $w \geq 1$

$$= 2\sum_{y=1}^{\infty} q^{w-1}p\; q^{w+y-1}p + q^{w-1}p\; q^{w-1}p = 2p^2q^{2w-2}\sum_{y-1}^{\infty} q^{y} + p^2q^{2w-2}$$

$$= \; p^2q^{2w-1}\,\frac{2}{1-q} + p^2q^{2w-2} = p^2q^{2w-2}\left(\frac{2q}{1-q} + 1\right) = p^2q^{2w-2}\frac{1+q}{1-q}$$

$$= pq^{2w-2}(1+q)$$

Multiple alternatives for counter-examples for (iv)

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