Admissions Testing Service

Sixth Term Examination Papers

9475

MATHEMATICS 3

Morning

FRIDAY 19 JUNE 2015

Time: 3 hours

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Please read this page carefully, but do not open this question paper until you are told that you may do so.

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Please wait to be told you may begin before turning this page.

This question paper consists of 9 printed pages and 3 blank pages.

Section A: Pure Mathematics

1 **(i)** Let

$$I_n = \int_0^\infty \frac{1}{(1+u^2)^n}\,\mathrm{d}u\,,$$

where n is a positive integer. Show that

$$I_n - I_{n+1} = \frac{1}{2n} I_n$$

and deduce that

$$I_{n+1} = \frac{(2n)!\,\pi}{2^{2n+1}(n!)^2}\,.$$

(ii) Let

$$J = \int_0^\infty \mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x\,,$$

where f is any function for which the integral exists. Show that

$$J = \int_0^\infty x^{-2}\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x \;=\; \frac{1}{2}\int_0^\infty (1+x^{-2})\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x \;=\; \int_0^\infty \mathrm{f}\big(u^2\big)\,\mathrm{d}u\,.$$

(iii) Hence evaluate

$$\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}\,\mathrm{d}x\,,$$

where n is a positive integer.

2 If s_1, s_2, s_3, ... and t_1, t_2, t_3, ... are sequences of positive numbers, we write

$$(s_n) \leqslant (t_n)$$

to mean

"there exists a positive integer m such that $s_n \leqslant t_n$ whenever $n \geqslant m$".

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate m; in the case of a false statement, you should give a counterexample.

(i) $(1000n) \leqslant (n^2)$.

(ii) If it is not the case that $(s_n) \leqslant (t_n)$, then it is the case that $(t_n) \leqslant (s_n)$.

(iii) If $(s_n) \leqslant (t_n)$ and $(t_n) \leqslant (u_n)$, then $(s_n) \leqslant (u_n)$.

(iv) $(n^2) \leqslant (2^n)$.

3 In this question, r and θ are polar coordinates with $r \geqslant 0$ and $-\pi < \theta \leqslant \pi$, and a and b are positive constants.

Let L be a fixed line and let A be a fixed point not lying on L. Then the locus of points that are a fixed distance (call it d) from L measured along lines through A is called a *conchoid of Nicomedes.*

(i) Show that if

$$|r - a\sec\theta| = b\,, \qquad (*)$$

where $a > b$, then $\sec\theta > 0$. Show that all points with coordinates satisfying $(*)$ lie on a certain conchoid of Nicomedes (you should identify L, d and A). Sketch the locus of these points.

(ii) In the case $a < b$, sketch the curve (including the loop for which $\sec\theta < 0$) given by

$$|r - a\sec\theta| = b\,.$$

Find the area of the loop in the case $a = 1$ and $b = 2$.

[Note: $\int \sec\theta \,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C$.]

4 **(i)** If a, b and c are all real, show that the equation

$$z^3 + az^2 + bz + c = 0 \qquad (*)$$

has at least one real root.

(ii) Let

$$S_1 = z_1 + z_2 + z_3, \qquad S_2 = z_1^2 + z_2^2 + z_3^2, \qquad S_3 = z_1^3 + z_2^3 + z_3^3\,,$$

where z_1, z_2 and z_3 are the roots of the equation $(*)$. Express a and b in terms of S_1 and S_2, and show that

$$6c = -S_1^3 + 3S_1S_2 - 2S_3\,.$$

(iii) The six real numbers r_k and θ_k $(k = 1,\ 2,\ 3)$, where $r_k > 0$ and $-\pi < \theta_k < \pi$, satisfy

$$\sum_{k=1}^{3} r_k \sin(\theta_k) = 0\,, \qquad \sum_{k=1}^{3} r_k^2 \sin(2\theta_k) = 0\,, \qquad \sum_{k=1}^{3} r_k^3 \sin(3\theta_k) = 0\,.$$

Show that $\theta_k = 0$ for at least one value of k.

Show further that if $\theta_1 = 0$ then $\theta_2 = -\theta_3$.

5 **(i)** In the following argument to show that $\sqrt{2}$ is irrational, give proofs appropriate for steps 3, 5 and 6.

1. Assume that $\sqrt{2}$ is rational.
2. Define the set S to be the set of positive integers with the following property:

   n is in S if and only if $n\sqrt{2}$ is an integer.
3. Show that the set S contains at least one positive integer.
4. Define the integer k to be the smallest positive integer in S.
5. Show that $(\sqrt{2} - 1)k$ is in S.
6. Show that steps 4 and 5 are contradictory and hence that $\sqrt{2}$ is irrational.

(ii) Prove that $2^{\frac{1}{3}}$ is rational if and only if $2^{\frac{2}{3}}$ is rational.

Use an argument similar to that of part (i) to prove that $2^{\frac{1}{3}}$ and $2^{\frac{2}{3}}$ are irrational.

6 **(i)** Let w and z be complex numbers, and let $u = w + z$ and $v = w^2 + z^2$. Prove that w and z are real if and only if u and v are real and $u^2 \leqslant 2v$.

(ii) The complex numbers u, w and z satisfy the equations

$$\begin{aligned} w + z - u &= 0 \\ w^2 + z^2 - u^2 &= -\tfrac{2}{3} \\ w^3 + z^3 - \lambda u &= -\lambda \end{aligned}$$

where λ is a positive real number. Show that for all values of λ except one (which you should find) there are three possible values of u, all real.

Are w and z necessarily real? Give a proof or counterexample.

7 An operator D is defined, for any function f, by

$$\mathrm{Df}(x) = x\frac{\mathrm{df}(x)}{\mathrm{d}x}.$$

The notation D^n means that D is applied n times; for example

$$\mathrm{D}^2\mathrm{f}(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{df}(x)}{\mathrm{d}x}\right).$$

Show that, for any constant a, $\mathrm{D}^2x^a = a^2x^a$.

(i) Show that if $\mathrm{P}(x)$ is a polynomial of degree r (where $r \geqslant 1$) then, for any positive integer n, $\mathrm{D}^n\mathrm{P}(x)$ is also a polynomial of degree r.

(ii) Show that if n and m are positive integers with $n < m$, then $\mathrm{D}^n(1-x)^m$ is divisible by $(1-x)^{m-n}$.

(iii) Deduce that, if m and n are positive integers with $n < m$, then

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^n = 0.$$

8 **(i)** Show that under the changes of variable $x = r\cos\theta$ and $y = r\sin\theta$, where r is a function of θ with $r > 0$, the differential equation

$$(y+x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$$

becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r = 0\,.$$

Sketch a solution in the x-y plane.

(ii) Show that the solutions of

$$\big(y + x - x(x^2+y^2)\big)\ \frac{\mathrm{d}y}{\mathrm{d}x} = y - x - y(x^2+y^2)$$

can be written in the form

$$r^2 = \frac{1}{1 + A\mathrm{e}^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of A.

Section B: Mechanics

9 A particle P of mass m moves on a smooth fixed straight horizontal rail and is attached to a fixed peg Q by a light elastic string of natural length a and modulus λ. The peg Q is a distance a from the rail. Initially P is at rest with $PQ = a$.

An impulse imparts to P a speed v along the rail. Let x be the displacement at time t of P from its initial position. Obtain the equation

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{x^2 + a^2} - a\right)^2$$

where $k^2 = \lambda/(ma)$, $k > 0$ and the dot denotes differentiation with respect to t.

Find, in terms of k, a and v, the greatest value, x_0, attained by x. Find also the acceleration of P at $x = x_0$.

Obtain, in the form of an integral, an expression for the period of the motion. Show that, in the case $v \ll ka$ (that is, v is much less than ka), this is approximately

$$\sqrt{\frac{32a}{kv}} \int_0^1 \frac{1}{\sqrt{1 - u^4}}\,\mathrm{d}u\,.$$

10 A light rod of length $2a$ has a particle of mass m attached to each end and it moves in a vertical plane. The midpoint of the rod has coordinates (x, y), where the x-axis is horizontal (within the plane of motion) and y is the height above a horizontal table. Initially, the rod is vertical, and at time t later it is inclined at an angle θ to the vertical.

Show that the velocity of one particle can be written in the form

$$\begin{pmatrix} \dot{x} + a\dot{\theta}\cos\theta \\ \dot{y} - a\dot{\theta}\sin\theta \end{pmatrix}$$

and that

$$m\begin{pmatrix} \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

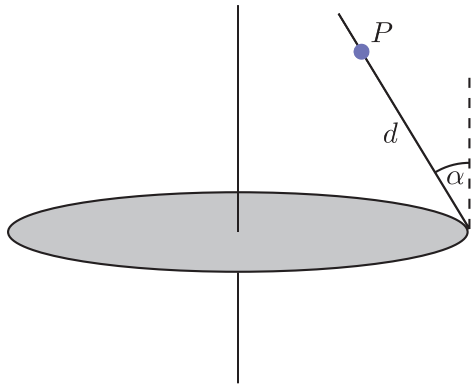
where the dots denote differentiation with respect to time t and T is the tension in the rod. Obtain the corresponding equations for the other particle.

Deduce that $\ddot{x} = 0$, $\ddot{y} = -g$ and $\ddot{\theta} = 0$.

Initially, the midpoint of the rod is a height h above the table, the velocity of the higher particle is $\begin{pmatrix} u \\ v \end{pmatrix}$, and the velocity of the lower particle is $\begin{pmatrix} 0 \\ v \end{pmatrix}$. Given that the two particles hit the table for the first time simultaneously, when the rod has rotated by $\frac{1}{2}\pi$, show that

$$2hu^2 = \pi^2 a^2 g - 2\pi uva\,.$$

11 **(i)** A horizontal disc of radius r rotates about a vertical axis through its centre with angular speed ω. One end of a light rod is fixed by a smooth hinge to the edge of the disc so that it can rotate freely in a vertical plane through the centre of the disc. A particle P of mass m is attached to the rod at a distance d from the hinge. The rod makes a constant angle α with the upward vertical, as shown in the diagram, and $d \sin \alpha < r$.



By considering moments about the hinge for the (light) rod, show that the force exerted on the rod by P is parallel to the rod.

Show also that

$$r \cot \alpha = a + d \cos \alpha \,,$$

where $a = \dfrac{g}{\omega^2}$. State clearly the direction of the force exerted by the hinge on the rod, and find an expression for its magnitude in terms of m, g and α.

(ii) The disc and rod rotate as in part (i), but two particles (instead of P) are attached to the rod. The masses of the particles are m_1 and m_2 and they are attached to the rod at distances d_1 and d_2 from the hinge, respectively. The rod makes a constant angle β with the upward vertical and $d_1 \sin \beta < d_2 \sin \beta < r$. Show that β satisfies an equation of the form

$$r \cot \beta = a + b \cos \beta \,,$$

where b should be expressed in terms of d_1, d_2, m_1 and m_2.

Section C: Probability and Statistics

12 A 6-sided fair die has the numbers 1, 2, 3, 4, 5, 6 on its faces. The die is thrown n times, the outcome (the number on the top face) of each throw being independent of the outcome of any other throw. The random variable S_n is the sum of the outcomes.

(i) The random variable R_n is the remainder when S_n is divided by 6. Write down the probability generating function, $\mathrm{G}(x)$, of R_1 and show that the probability generating function of R_2 is also $\mathrm{G}(x)$. Use a generating function to find the probability that S_n is divisible by 6.

(ii) The random variable T_n is the remainder when S_n is divided by 5. Write down the probability generating function, $\mathrm{G}_1(x)$, of T_1 and show that $\mathrm{G}_2(x)$, the probability generating function of T_2, is given by

$$\mathrm{G}_2(x) = \tfrac{1}{36}(x^2 + 7y)$$

where $y = 1 + x + x^2 + x^3 + x^4$.

Obtain the probability generating function of T_n and hence show that the probability that S_n is divisible by 5 is

$$\frac{1}{5}\left(1 - \frac{1}{6^n}\right)$$

if n is not divisible by 5. What is the corresponding probability if n is divisible by 5?

13 Each of the two independent random variables X and Y is uniformly distributed on the interval $[0, 1]$.

(i) By considering the lines $x + y =$ constant in the x-y plane, find the cumulative distribution function of $X + Y$.

Hence show that the probability density function f of $(X + Y)^{-1}$ is given by

$$\mathrm{f}(t) = \begin{cases} 2t^{-2} - t^{-3} & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \\ t^{-3} & \text{for } 1 \leqslant t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate $\mathrm{E}\left(\dfrac{1}{X+Y}\right)$.

(ii) Find the cumulative distribution function of Y/X and use this result to find the probability density function of $\dfrac{X}{X+Y}$.

Write down $\mathrm{E}\left(\dfrac{X}{X+Y}\right)$ and verify your result by integration.

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