Admissions Testing Service

Sixth Term Examination Papers **9475**

MATHEMATICS 3 Morning

WEDNESDAY 26 JUNE 2013 Time: 3 hours

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Please read this page carefully, but do not open this question paper until you are told that you may do so.

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Please wait to be told you may begin before turning this page.

This question paper consists of 10 printed pages and 2 blank pages.

Section A: Pure Mathematics

1 Given that $t = \tan \frac{1}{2}x$, show that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}(1+t^2)$ and $\sin x = \dfrac{2t}{1+t^2}$.

Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\,\mathrm{d}x = \frac{2}{\sqrt{1-a^2}} \arctan \frac{\sqrt{1-a}}{\sqrt{1+a}} \qquad (0 < a < 1).$$

Let

$$I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^n x}{2+\sin x}\,\mathrm{d}x \qquad (n \geqslant 0).$$

By considering $I_{n+1} + 2I_n$, or otherwise, evaluate I_3.

2 In this question, you may ignore questions of convergence.

Let $y = \dfrac{\arcsin x}{\sqrt{1-x^2}}$. Show that

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy - 1 = 0$$

and prove that, for any positive integer n,

$$(1-x^2)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+3)x\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} - (n+1)^2\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0\,.$$

Hence obtain the Maclaurin series for $\dfrac{\arcsin x}{\sqrt{1-x^2}}$, giving the general term for odd and for even powers of x.

Evaluate the infinite sum

$$1 + \frac{1}{3!} + \frac{2^2}{5!} + \frac{2^2\times 3^2}{7!} + \cdots + \frac{2^2\times 3^2\times\cdots\times n^2}{(2n+1)!} + \cdots\,.$$

3 The four vertices P_i $(i = 1, 2, 3, 4)$ of a regular tetrahedron lie on the surface of a sphere with centre at O and of radius 1. The position vector of P_i with respect to O is $\mathbf{p}_i$ $(i = 1, 2, 3, 4)$. Use the fact that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$ to show that $\mathbf{p}_i \,.\, \mathbf{p}_j = -\frac{1}{3}$ for $i \neq j$.

Let X be any point on the surface of the sphere, and let XP_i denote the length of the line joining X and P_i $(i = 1, 2, 3, 4)$.

(i) By writing $(XP_i)^2$ as $(\mathbf{p}_i - \mathbf{x}) \,.\, (\mathbf{p}_i - \mathbf{x})$, where $\mathbf{x}$ is the position vector of X with respect to O, show that

$$\sum_{i=1}^{4} (XP_i)^2 = 8\,.$$

(ii) Given that P_1 has coordinates $(0, 0, 1)$ and that the coordinates of P_2 are of the form $(a, 0, b)$, where $a > 0$, show that $a = 2\sqrt{2}/3$ and $b = -1/3$, and find the coordinates of P_3 and P_4.

(iii) Show that

$$\sum_{i=1}^{4} (XP_i)^4 = 4 \sum_{i=1}^{4} (1 - \mathbf{x} \,.\, \mathbf{p}_i)^2\,.$$

By letting the coordinates of X be (x, y, z), show further that $\sum\limits_{i=1}^{4} (XP_i)^4$ is independent of the position of X.

4 Show that $(z - \mathrm{e}^{\mathrm{i}\theta})(z - \mathrm{e}^{-\mathrm{i}\theta}) = z^2 - 2z\cos\theta + 1\,.$

Write down the $(2n)$th roots of -1 in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^{n} \left(z^2 - 2z\cos\left(\tfrac{(2k-1)\pi}{2n}\right) + 1 \right)\,.$$

Here, n is a positive integer, and the $\prod$ notation denotes the product.

(i) By substituting $z = \mathrm{i}$ show that, when n is even,

$$\cos\left(\tfrac{\pi}{2n}\right) \cos\left(\tfrac{3\pi}{2n}\right) \cos\left(\tfrac{5\pi}{2n}\right) \cdots \cos\left(\tfrac{(2n-1)\pi}{2n}\right) = (-1)^{\frac{1}{2}n} 2^{1-n}\,.$$

(ii) Show that, when n is odd,

$$\cos^2\left(\tfrac{\pi}{2n}\right) \cos^2\left(\tfrac{3\pi}{2n}\right) \cos^2\left(\tfrac{5\pi}{2n}\right) \cdots \cos^2\left(\tfrac{(n-2)\pi}{2n}\right) = n2^{1-n}\,.$$

You may use without proof the fact that $1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when n is odd.

5 In this question, you may assume that, if a, b and c are positive integers such that a and b are coprime and a divides bc, then a divides c. (Two positive integers are said to be *coprime* if their highest common factor is 1.)

(i) Suppose that there are positive integers p, q, n and N such that p and q are coprime and $q^n N = p^n$. Show that $N = kp^n$ for some positive integer k and deduce the value of q.

Hence prove that, for any positive integers n and N, $\sqrt[n]{N}$ is either a positive integer or irrational.

(ii) Suppose that there are positive integers a, b, c and d such that a and b are coprime and c and d are coprime, and $a^a d^b = b^a c^b$. Prove that $d^b = b^a$ and deduce that, if p is a prime factor of d, then p is also a prime factor of b.

If p^m and p^n are the highest powers of the prime number p that divide d and b, respectively, express b in terms of a, m and n and hence show that $p^n \leqslant n$. Deduce the value of b. (You may assume that if $x > 0$ and $y \geqslant 2$ then $y^x > x$.)

Hence prove that, if r is a positive rational number such that r^r is rational, then r is a positive integer.

6 Let z and w be complex numbers. Use a diagram to show that $|z - w| \leqslant |z| + |w|$.

For any complex numbers z and w, E is defined by

$$E = zw^* + z^*w + 2|zw|\,.$$

(i) Show that $|z - w|^2 = (|z| + |w|)^2 - E$, and deduce that E is real and non-negative.

(ii) Show that $|1 - zw^*|^2 = (1 + |zw|)^2 - E$.

Hence show that, if both $|z| > 1$ and $|w| > 1$, then

$$\frac{|z - w|}{|1 - zw^*|} \leqslant \frac{|z| + |w|}{1 + |zw|}\,.$$

Does this inequality also hold if both $|z| < 1$ and $|w| < 1$?

7 **(i)** Let $y(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$ with $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \tfrac{1}{2}y^4 .$$

Show by differentiation that E is constant and deduce that $|y(x)| \leqslant 1$ for all x.

(ii) Let $v(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}v}{\mathrm{d}x} + \sinh v = 0$ with $v = \ln 3$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + 2\cosh v .$$

Show that $\dfrac{\mathrm{dE}}{\mathrm{d}x} \leqslant 0$ for $x \geqslant 0$ and deduce that $\cosh v(x) \leqslant \frac{5}{3}$ for $x \geqslant 0$.

(iii) Let $w(x)$ be a solution of the differential equation

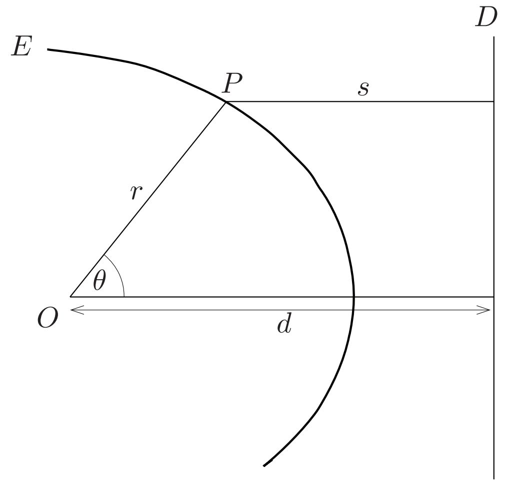
$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + (5\cosh x - 4\sinh x - 3)\frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w) = 0$$

with $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \dfrac{1}{\sqrt{2}}$ and $w = 0$ at $x = 0$. Show that $\cosh w(x) \leqslant \frac{5}{4}$ for $x \geqslant 0$.

 [Turn over

8 Evaluate $\displaystyle\sum_{r=0}^{n-1} \mathrm{e}^{2\mathrm{i}(\alpha + r\pi/n)}$ where α is a fixed angle and $n \geqslant 2$.

The fixed point O is a distance d from a fixed line D. For any point P, let s be the distance from P to D and let r be the distance from P to O. Write down an expression for s in terms of d, r and the angle θ, where θ is as shown in the diagram below.



The curve E shown in the diagram is such that, for any point P on E, the relation $r = ks$ holds, where k is a fixed number with $0 < k < 1$.

Each of the n lines $L_1, \ldots, L_n$ passes through O and the angle between adjacent lines is $\frac{\pi}{n}$. The line L_j $(j = 1, \ldots, n)$ intersects E in two points forming a chord of length l_j. Show that, for $n \geqslant 2$,

$$\sum_{j=1}^{n} \frac{1}{l_j} = \frac{(2 - k^2)n}{4kd}\,.$$

Section B: Mechanics

9 A sphere of radius R and uniform density $\rho_{\rm s}$ is floating in a large tank of liquid of uniform density ρ. Given that the centre of the sphere is a distance x above the level of the liquid, where $x < R$, show that the volume of liquid displaced is

$$\frac{\pi}{3}(2R^3 - 3R^2x + x^3)\,.$$

The sphere is acted upon by two forces only: its weight and an upward force equal in magnitude to the weight of the liquid it has displaced. Show that

$$4R^3\rho_{\rm s}(g + \ddot{x}) = (2R^3 - 3R^2x + x^3)\rho g\,.$$

Given that the sphere is in equilibrium when $x = \frac{1}{2}R$, find $\rho_{\rm s}$ in terms of ρ. Find, in terms of R and g, the period of small oscillations about this equilibrium position.

10 A uniform rod AB has mass M and length $2a$. The point P lies on the rod a distance $a - x$ from A. Show that the moment of inertia of the rod about an axis through P and perpendicular to the rod is

$$\tfrac{1}{3}M(a^2 + 3x^2)\,.$$

The rod is free to rotate, in a horizontal plane, about a fixed vertical axis through P. Initially the rod is at rest. The end B is struck by a particle of mass m moving horizontally with speed u in a direction perpendicular to the rod. The coefficient of restitution between the rod and the particle is e. Show that the angular velocity of the rod immediately after impact is

$$\frac{3mu(1+e)(a+x)}{M(a^2+3x^2)+3m(a+x)^2}\,.$$

In the case $m = 2M$, find the value of x for which the angular velocity is greatest and show that this angular velocity is $u(1+e)/a$.

11 An equilateral triangle, comprising three light rods each of length $\sqrt{3}a$, has a particle of mass m attached to each of its vertices. The triangle is suspended horizontally from a point vertically above its centre by three identical springs, so that the springs and rods form a tetrahedron. Each spring has natural length a and modulus of elasticity kmg, and is light. Show that when the springs make an angle θ with the horizontal the tension in each spring is

$$\frac{kmg(1-\cos\theta)}{\cos\theta}\,.$$

Given that the triangle is in equilibrium when $\theta = \frac{1}{6}\pi$, show that $k = 4\sqrt{3}+6$.

The triangle is released from rest from the position at which $\theta = \frac{1}{3}\pi$. Show that when it passes through the equilibrium position its speed V satisfies

$$V^2 = \frac{4ag}{3}(6+\sqrt{3})\,.$$

Section C: Probability and Statistics

12 A list consists only of letters A and B arranged in a row. In the list, there are a letter As and b letter Bs, where $a \geqslant 2$ and $b \geqslant 2$, and $a + b = n$. Each possible ordering of the letters is equally probable. The random variable X_1 is defined by

$$X_1 = \begin{cases} 1 & \text{if the first letter in the row is } A; \\ 0 & \text{otherwise.} \end{cases}$$

The random variables X_k $(2 \leqslant k \leqslant n)$ are defined by

$$X_k = \begin{cases} 1 & \text{if the } (k-1)\text{th letter is } B \text{ and the } k\text{th is } A; \\ 0 & \text{otherwise.} \end{cases}$$

The random variable S is defined by $S = \sum\limits_{i=1}^{n} X_i$.

(i) Find expressions for $\mathrm{E}(X_i)$, distinguishing between the cases $i = 1$ and $i \neq 1$, and show that $\mathrm{E}(S) = \dfrac{a(b+1)}{n}$.

(ii) Show that:

(a) for $j \geqslant 3$, $\mathrm{E}(X_1 X_j) = \dfrac{a(a-1)b}{n(n-1)(n-2)}$;

(b) $\sum\limits_{i=2}^{n-2} \Bigl(\sum\limits_{j=i+2}^{n} \mathrm{E}(X_i X_j) \Bigr) = \dfrac{a(a-1)b(b-1)}{2n(n-1)}$;

(c) $\mathrm{Var}\,(S) = \dfrac{a(a-1)b(b+1)}{n^2(n-1)}$.

13 **(a)** The continuous random variable X satisfies $0 \leqslant X \leqslant 1$, and has probability density function $\mathrm{f}(x)$ and cumulative distribution function $\mathrm{F}(x)$. The greatest value of $\mathrm{f}(x)$ is M, so that $0 \leqslant \mathrm{f}(x) \leqslant M$.

(i) Show that $0 \leqslant \mathrm{F}(x) \leqslant Mx$ for $0 \leqslant x \leqslant 1$.

(ii) For any function $\mathrm{g}(x)$, show that

$$\int_0^1 2\mathrm{g}(x)\mathrm{F}(x)\mathrm{f}(x)\mathrm{d}x = \mathrm{g}(1) - \int_0^1 \mathrm{g}'(x)\big(\mathrm{F}(x)\big)^2\mathrm{d}x\,.$$

(b) The continuous random variable Y satisfies $0 \leqslant Y \leqslant 1$, and has probability density function $k\mathrm{F}(y)\mathrm{f}(y)$, where f and F are as above.

(i) Determine the value of the constant k.

(ii) Show that

$$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \leqslant \mathrm{E}(Y^n) \leqslant 2M\mu_{n+1}\,,$$

where $\mu_{n+1} = \mathrm{E}(X^{n+1})$ and $n \geqslant 0$.

(iii) Hence show that, for $n \geqslant 1$,

$$\mu_n \geqslant \frac{n}{(n+1)M} - \frac{n-1}{n+1}\,.$$

BLANK PAGE

BLANK PAGE