Cambridge Assessment
Admissions Testing

Sixth Term Examination Paper [STEP]

Mathematics 2 [9470]

2023

Examiners' Report

Mark Scheme

STEP MATHEMATICS 2

2023

Examiners' Report

STEP 2 Introduction

Many candidates were able to express their reasoning clearly and presented good solutions to the questions that they attempted. There were excellent solutions seen for all of the questions.

An area where candidates struggled in several questions was in the direction of the logic that was required in a solution. Some candidates failed to appreciate that separate arguments may be needed for the "if" and "only if" parts of a question and, in some cases, candidates produced correct arguments, but for the wrong direction.

In several questions it was clear that candidates who used sketches or diagrams generally performed much better that those who did not. Sketches often also helped to make the solution clearer and easier to understand.

Several questions on the STEP papers ask candidates to show a given result. Candidates should be aware that there is a need to present sufficient detail in their solutions so that it is clear that the reasoning is well understood.

Question 1

The first part of this question was often completed well, although candidates should note that in questions where the result is given it is important to show enough detail in the solution. Weaker candidates failed to change the limits or did not differentiate $1/x$ correctly when completing the substitution.

Most candidates realised that part (ii)(a) could be completed by applying the result from part (i) and were able to select the correct values for a and b. However, many did not realise that the result from part (i) was not directly applicable to part (ii)(b) and so did not gain any marks for that part, although some candidates did realise that the answer of zero could not be correct and received some credit for recognising that the function was even and so could identify the start of a correct solution. Solutions that applied the result from part (i) successfully often achieved full marks, although in some cases the way in which limits were dealt with was not sufficient. A significant number of candidates recognised that part (ii)(b) could be solved with a tan substitution and while this approach was successful, in some cases the final answer was not written in its simplest form.

In part (iii)(a) many candidates recognised that the same substitution would produce the required results, but as in part (i) several cases did not produce clear enough solutions to earn all of the marks. Most candidates were able to successfully calculate the value of the integral. Many candidates did not choose a suitable substitution for part (iii)(b), but those who did generally managed to reach an appropriate form of the integral that could be compared to the original. Many then deduced the correct answer from this, but several did not recognise the significance of the new integral and then attempted other substitutions with little success.

Question 2

In part (i) most attempts to show $\tan\alpha = \tan 8\alpha$ were successful and included sufficient detail to earn the marks. Some candidates attempted to use the half-angle formula instead of the double-angle formula. This does not work, as the logic goes in the wrong direction, and leads to a quadratic with two solutions following which candidates simply asserted that the half angle is the correct solution. A large proportion of students made no further progress on this question.

Of the students that did progress further on this part, many became confused by the restrictions on range. They realised solutions were of the form $(\tan\alpha, \tan 2\alpha, \tan 4\alpha)$, but then tried to simultaneously have α, 2α, and 4α between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. These solutions erroneously discarded (or did not even find) solutions other than (0, 0, 0). However, some students did realise that they could subtract or add multiples of π to some of the arguments until all were in the required range. These attempts often obtained full marks for this part. Some attempts used an alternative method, rather than using periodicity of tan to solve $\tan\alpha = \tan 8\alpha$, they rewrote in terms of sin and cos and used addition formulae to obtain $\sin 7\alpha = 0$. However, these attempts often only checked the logic in one direction and did not comment that $\cos\alpha$ and $\cos 8\alpha$ were non-zero in these cases.

In part (ii) a good number of students attempted the substitution $x = \tan\alpha$, and many of these either quoted or proved the triple angle formula for tan. Again, some made no further progress (often scripts that attempted both (i) and (ii) made similar amounts of progress on both parts). Since this part asked for the number of solutions, rather than finding all solutions, many students only found solutions for x. This lost credit unless it was accompanied by a check that each value of x led to a value of y and z. Several candidates failed to discard $x = \pm\frac{\pi}{2}$, for which tan(x) is undefined, leading to an answer of 27, rather than 25.

In part (iii) (a) many students attempted a useful trigonometric substitution in this part (either cos or sin). Most scripts that attempted a useful substitution made correct use of the double angle formula to arrive at $\cos\alpha = \cos 8\alpha$ or a similar equation using a sin substitution. Solving the equation cos(α) = cos(8α) gave many students significant difficulty. Many attempts used only the periodicity of cos, and not the evenness, thus only obtaining half of the solutions. Others failed to restrict to a range where cos is single-valued, thus finding the same solution for x for different values of α and erroneously counting these as different solutions. Those who chose to draw a sketch of the graph to aid their thinking generally produced better solutions in this part.

In part (iii) (b) most attempts found the correct octic. Attempts that had found fewer than 8 solutions in (a) often made no further progress. Candidates who had found 8 solutions in (a) often obtained full marks in this part.

Question 3

In part (i) (a) when assuming that the degree of p is odd for a contradiction, many also assumed that the lead coefficient of p(x) was positive and so made the statement that p(x) tends to minus infinity as x tends to minus infinity which is not necessarily correct (unless an argument that the lead coefficient is positive was provided). Many candidates did not provide sufficient detail and so were not awarded full marks for this part.

In part (i)(b) candidates generally produced good answers, but a number lost marks for not stating that the $(n+1)^{th}$ derivative of p is zero sufficiently clearly. Some used +... at the end of the sum of polynomials that define q(x) or just didn't discuss the final term of q'(x) and again in these cases it was not sufficiently clear that the key idea had been understood.

In part (ii) (a) candidates generally completed the first part well, but a significant number of candidates lost a mark because their argument was the wrong way round, arguing that B implies A rather than A implies B. A significant number of candidates realised that all the stationary points of q must have a positive y-coordinate but they didn't link this to q(x) being positive for large $|x|$ to get all the marks.

In part (ii) (b) the first part was again usually very well done. In a similar way to part (a) there were a good number of impressive answers to 'q(x)>0 for all x' but many lost marks by not providing sufficient detail or not including all aspects of the argument (particularly that q(x)>0 for large x).

Part (ii) (c) was generally very well done. Virtually all candidates used the right method for the first part but some lost a mark for not providing sufficient detail in the substitution in integration by parts. Most did the rest of this part well but quite a few candidates lost marks for not dealing with the end term of the summation correctly - in the main line of the solution it is an integral which candidates should explain is zero. Some neglected to include this term without comment or used +... at the end of the sum and, in these cases, it was not clear that the idea had been understood.

Question 4

There were a wide variety of different approaches to part (i), including some which identified what the four roots of a quartic with integer coefficients would have to be in order for the required condition to be met.

In part (ii) many candidates were able to identify a valid approach to the question although some algebraic errors meant that some did not reach the correct final polynomial. As with part (i) there were a number of different approaches that were taken.

In part (iii) most candidates recognised that a translation of the graph would provide a cubic with the correct roots. Many were then able to apply similar methods to the earlier parts of the question to obtain the required polynomial with integer coefficients.

Many candidates did not attempt the final part of the question, but those who did were generally able to adapt the methods from the previous parts successfully to make good progress.

Question 5

In part (i) (a) most candidates realised that induction was necessary. Although "explain briefly" was written in the question, some candidates omitted necessary components of an inductive argument here. Some candidates incorrectly stated that the sequence always increased. A popular alternative method was stating $x_n+2 > x_n+1$. In this case it is necessary to observe that the denominator is positive to secure full marks.

In part (i) (b) many candidates were successful here in rewriting x_{n+1}^2-2 in terms of x_n but some failed to assert (and very briefly justify) the strict positivity of $(x_n+1)^2$ in order to show that x_{n+1}^2-2 and x_n^2-2 have opposite signs. When showing $|x_{n+1}^2-2| \leq |x_n^2-2|/4$ the most common mistake was to not use absolute value signs, and write false assertions like $x_{n+1}^2-2 \leq (x_n^2-2)/4$, which is false for odd n.

In part (i) (c) many students used the inequality in the previous part repeatedly to write $|x_{10}^2-2| \leq |x_0^2-2|/4^{10}$ but did not give a justification that $4^{10}>10^6$. A small number of candidates were able to calculate x_{10}, and x_{10}^2 successfully and numerically compare these to 2 and $2-10^{-6}$, however almost all attempts at this were unsuccessful.

Almost all candidates who attempted part (ii) (a) earned at least one mark. In several cases candidates did not formulate a standard inductive argument, either missing the base case or not using an inductive hypothesis.

In part (ii) (b) many candidates used n=0 as a base case, but this is not valid here. Of those who opted for an alternative method of using recursion to write $y_n - \sqrt{2}$ in terms of $y_0 - \sqrt{2}$, few were able to justify the exponent for powers of 2. Candidates who attempted a full inductive proof often earned at least 2 of the 4 marks for this part.

Candidates attempting part (ii)(c) often earned some marks for showing the correct method, but errors in the accuracy of the work meant that few were able to achieve full marks here.

Question 6

Most candidates were able to complete the proof by induction on which the other parts of the question are based. In some cases, the matrix multiplication was not completed correctly (such as calculating the product **AB** rather than **BA**). Throughout the question some candidates also got confused about the different variables involved although in some cases where this was clearly simply a mislabelling, they were given the benefit of the doubt.

Most candidates were able to see how the relevant matrices could be used to obtain answers for both part (i) and part (ii), but in a small number of cases there was insufficient justification to show that the way in which the result was deduced had been understood.

In part (iii) most candidates were able to show that $\mathbf{Q}^2 = \mathbf{I} + \mathbf{Q}$, but many candidates were unable to make much more progress from this point. There were a small number of excellent solutions, carefully checking all of the relevant cases in each part and providing very clear explanations of the reasoning.

Question 7

Most candidates were successful in the first two parts, with marks being lost mostly due to the small inaccuracy of forgetting the square root in the expression for the modulus of a complex number.

Part (iii) was also typically done well, with most candidates picking up the idea of dividing by 5, however with mixed accuracy on the other factor. The candidates who picked up that the other factor can be written as a sum of squares were mostly successful in this part, as were almost all the candidates who attempted part (iv).

Parts (v) and (vi) discriminated between candidates, with many successfully getting through (i)-(iv) with full marks but unfortunately making little to no progress on these two. Many failed to spot the decompositions $1001^2 + 9^2$ in (v) and $1001^2 + 6^2$ in (vi). The candidates who found these got access to the marks, though many didn’t manage to find three solutions in part (v). This was from either overlooking the Pythagorean triple of (3, 4, 5) or the simpler solution obtained by noting that 25 is the square of 5. In part (vi), many candidates either chose the wrong complex number and did not try another one or by failed to notice that 10028 or 2943 are divisible by 109.

Question 8

In part (i), most candidates answered the “only if” direction of the argument successfully, often with a diagram. Many candidates did not realise they needed to prove “if” separately, but those that did usually answered this well. Most students wrote a list of simultaneous equations in the edge lengths to solve. Appeals to symmetry were accepted without much detail needed. Many candidates did not attempt the later parts of the question.

Part (ii) was generally answered well. Many students attempted the cosine rule, which required some detail relating it to the given problem. A surprising number of candidates would set the direction vectors of each edge equal, rather than just their lengths. Some ended up confusing scalars and vectors due to poor notation.

In part (iii) many candidates found $\mathbf{a.g}$ etc. rather than $|\overrightarrow{AG}|$. This could be made to work but needed to be made relevant to the question to earn marks. Some candidates thought $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ were the components of a vector and attempted to use Pythagoras, which got no credit. Several candidates found $|\overrightarrow{AG}|^2$ etc. in a non-symmetric form and attempted to appeal to symmetry, which was not accepted.

Part (iv) proved to be quite tricky for most candidates. Many ignored the questions prompting entirely or failed to relate it to a relevant geometrical idea. Few candidates used cosine successfully and attained the final two marks. Several candidates stated that right angles were possible at the end.

Question 9

Less than half of the candidates produced an accurate diagram for this question, with many leaving off some forces, or making errors with the gravitational force by not including g. This had an impact on their ability to proceed with the question, and often those with poorly presented diagrams had sign errors in their force balance equations (for example, with tension in the wrong direction). Most seemed to understand how to calculate frictional force. The inclusion of friction on the trailer and not the truck clearly confused some candidates, causing many of the question parts to be inaccessible.

Many candidates seemed to struggle with the fact that 6 equations had to be dealt with, and so struggled to identify which variables to eliminate and how to eliminate them. Additionally, some candidates did not realise that some of the forces would take different values in the different cases being considered.

Part (i) was done quite well overall, although for the second part, a fair number of candidates showed each side was equal to some expression involving the other variables, which is valid but took much more time than the direction approach using the equation of motion for the truck.

Part (ii) (a) this part was done well in some cases, although less well than the previous part. Most candidates who attempted this part were able to show the upper inequality, but the lower one proved to be more difficult.

Most attempts to part (ii) (b) only achieved two of the marks available. Many candidates did not recognise that the half angle formula was useful here and so struggled to make progress on the question.

Question 10

Part (i) was answered well. Almost all candidates used the discriminant condition correctly, even if their quadratic contained an error. The final two marks in this part were trickier to achieve. Many candidates substituted two values for alpha and then solved for A, B. Of these, it was fairly common for them to use $\alpha = 0, \frac{1}{2}\pi$ which were excluded from the range being considered. Many missed that one can simply set the coefficients of each side to zero. Some treated α as a variable to be solved for rather than varied.

Part (ii) was found very tricky. Some candidates were able to guess or intuit that the safe zone should be the parabola considered in part (i), but almost no candidate gave a proper explanation as to why it was only these points that could be reached. A few considered the 2D parabola correctly but did not consider 3D. Some wondered about the presence of a "floor" at $z = 0$. This only caused an issue if the candidate thought we were only interested in points of intersection with this plane, possibly caused by imagining the problem in the context of artillery as it is often presented in schools. In this case they answered with a circle and received no credit.

Part (iii) was fairly tough for most candidates. Many candidates who could not find the equation of a circle/sphere would attempt an explanation in words, which was almost never sufficient to earn credit. It was possible for candidates to guess the correct centre and radius with no mathematical justification, but this earned no credit.

Many candidates who attempted part (iv) were able to complete it successfully.

Candidates who attempted part (v) were generally able to pick up at least one mark by appealing to earlier calculations. Most appreciated the need to introduce separate times for Q and P, or a time delay between the two.

Question 11

There were very few substantial attempts at this question overall.

In part (i) a large number of candidates incorrectly stated that $Y = pX_1 + qX_2$. However, there were several good responses to this question with many candidates obtaining the correct value for at least one of the mean and variance of Y.

Similarly, in part (ii) many candidates were able to compute the mean and variance of Z_1 correctly. However, several candidates only computed $\mathrm{P}(B = 1)$ when asked to justify that Z_1 is a binomial variable.

Candidates generally struggled with part (iii), often comparing the variance and the mean incorrectly for the two facts that were required to be shown.

Question 12

This was the more popular of the two "Probability and Statistics" questions and a larger number of substantial attempts was seen.

Part (i) was generally completed well although in some cases there was insufficient explanation that "$Y \leq t$" is equivalent to "$X_i \leq t$ for all i".

Many candidates successfully calculated the value of $m(n)$ for part (ii), but some only stated that $m(n)$ increases, rather than considering the value of the limit.

In part (iii) many candidates successfully showed the formula for $\mu(n)$. A number of candidates attempted to prove that $\mu(n)$ is increasing by differentiating with respect to n and showing that this is a positive quantity. However, none of these candidates were able to produce a fully correct version of this approach.

In part (iii) (b) most candidates were able to calculate $\mu(2)$ correctly, but then a number of errors were seen in the subsequent argument. Common errors were to fail to consider which choice of square root is appropriate and to omit to consider the effect of squaring on an inequality.

STEP MATHEMATICS 2

2023

Mark Scheme

Question			Answer	Mark
1	(i)		$\displaystyle\int_a^b \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx = \int_{a^{-1}}^{b^{-1}} \frac{1}{\left(1+\frac{1}{t^2}\right)^{\frac{3}{2}}} \cdot -\frac{1}{t^2}\,dt$	M1
			$\displaystyle = \int_{a^{-1}}^{b^{-1}} \frac{-t}{\left(1+t^2\right)^{\frac{3}{2}}}\,dt$	A1
				[2]
	(ii)	(a)	$\displaystyle\int_{\frac{1}{2}}^{2} \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx = \int_{2}^{\frac{1}{2}} \frac{-t}{\left(1+t^2\right)^{\frac{3}{2}}}\,dt$	M1
			$\displaystyle = \left[(1+t^2)^{-\frac{1}{2}}\right]_2^{\frac{1}{2}}$	M1
			$\displaystyle = \left(\frac{5}{4}\right)^{-\frac{1}{2}} - 5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$	A1
				[3]
		(b)	The integrand is even, so $\displaystyle\int_{-2}^{2} \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx = 2\int_{0}^{2} \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx$	M1
			$\displaystyle = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{2} \frac{2}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx$ which is of the form in the stem	M1
			$\displaystyle = \lim_{\varepsilon\to 0} \left[2(1+t^2)^{-\frac{1}{2}}\right]_{\frac{1}{\varepsilon}}^{\frac{1}{2}}$	M1
			Dealing with limiting process for the lower limit	E1
			$= \frac{4}{\sqrt{5}}$	A1
			$\displaystyle -\lim_{\varepsilon\to 0} \frac{2\varepsilon}{\sqrt{1+\varepsilon^2}}$	A1
			so the limit is zero and the integral $= \frac{4}{\sqrt{5}}$	E1
				[7]
	(iii)	(a)	$\displaystyle\int_{\frac{1}{2}}^{2} \frac{1}{\left(1+x^2\right)^{2}}\,dx = \int_{2}^{\frac{1}{2}} \frac{-t^2}{\left(1+t^2\right)^{2}}\,dt$	M1
			$\displaystyle = \int_{\frac{1}{2}}^{2} \frac{x^2}{\left(1+x^2\right)^{2}}\,dx$	A1

			so $\int_{\frac{1}{2}}^{2} \frac{1}{\left(1+x^2\right)^2}\,dx = \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1}{\left(1+x^2\right)^2}\,dx + \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{x^2}{\left(1+x^2\right)^2}\,dx$	**M1**
			$= \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1}{1+x^2}\,dx$	**A1**
			$= \frac{1}{2}\left(arctan\,2 - arctan\,\frac{1}{2}\right)$	**B1**
				[5]
		(b)	$\int_{\frac{1}{2}}^{2} \frac{1-x}{x\left(1+x^2\right)^{\frac{1}{2}}}\,dx = \int_{2}^{\frac{1}{2}} \frac{t-1}{\left(1+\frac{1}{t^2}\right)^{\frac{1}{2}}} \cdot -\frac{1}{t^2}\,dt$	**M1**
			$= \int_{2}^{\frac{1}{2}} \frac{1-t}{t\left(1+t^2\right)^{\frac{1}{2}}}\,dt$	**A1**
			$= -\int_{\frac{1}{2}}^{2} \frac{1-x}{x\left(1+x^2\right)^{\frac{1}{2}}}\,dx$, so the integral = 0.	**E1**
				[3]

Question			Answer	Mark
2	(i)		Let $x = \tan\alpha$. Then $y = \frac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha$	
			so $z = \tan 4\alpha$, and so $\tan\alpha = \tan 8\alpha$.	E1
			giving $8\alpha = \alpha + n\pi$, or $\alpha = \frac{1}{7}n\pi$, (for $n = -3$ to 3).	M1
			Solutions are: $(0,0,0)$, $\left(\tan\left(\frac{1}{7}\alpha\right), \tan\left(\frac{2}{7}\alpha\right), \tan\left(-\frac{3}{7}\alpha\right)\right)$	B1
			and cyclic permutations of the latter	A1
			and $\left(\tan\left(-\frac{1}{7}\alpha\right), \tan\left(-\frac{2}{7}\alpha\right), \tan\left(\frac{3}{7}\alpha\right)\right)$ and its cyclic permutations	A1
				[5]
	(ii)		$\tan 3\alpha = \frac{\frac{2\tan\alpha}{1-\tan^2\alpha}+\tan\alpha}{1-\frac{2\tan^2\alpha}{1-\tan^2\alpha}}$	M1
			$= \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$	A1
			Let $x = \tan\alpha$; then $y = \tan 3\alpha$, $z = \tan 9\alpha$, so $\tan 27\alpha = \tan\alpha$	M1
			giving $26\alpha = n\pi$	A1
			which has 25 solutions with distinct values of $\tan\alpha$ because n = 13 does not give a possible value of $\tan\alpha$.	A1
			Checking that for each finite value of x, the denominators of y and z are defined (i.e. checking 1-3x^2 is non-zero).	E1
				[6]
	(iii)	(a)	Let $x = \cos\alpha$	M1
			the restriction on $\|x\|$ means this is a complete parametrisation of solutions	E1
			Then, using $\cos 2\alpha = 2\cos^2\alpha - 1$, $\cos 8\alpha = \cos\alpha$	M1
			so $8\alpha = \alpha + 2m\pi$, or $8\alpha = -\alpha + 2n\pi$	M1
			so $7\alpha = 2m\pi$ or $9\alpha = 2n\pi$	A1
			with 4 (m = 0 to 3) + 5 (n = 0 to 4)	M1
			– 1 (for $\alpha = 0$ twice) = 8 distinct solutions	A1
				[7]
		(b)	y quadratic, so z quartic in x, so x satisfies an octic equation	B1
			which has at most 8 roots, so there are no larger solutions.	E1
				[2]

Question			Answer	Mark
3	(i)	(a)	An odd degree polynomial takes positive and negative values for large enough $\|x\|$.	B1
			The degree of q is equal to the degree of p, and the coefficient of x^n is positive, because each derivative has lower degree and cannot affect the coefficient of x^n.	M1
			So $q(x) > 0$ for large enough $\|x\|$.	A1
				[3]
		(b)	$q'(x) = \sum_{k=0}^{n} p^{(k+1)}(x) = \sum_{k=0}^{n-1} p^{(k+1)}(x)$ [$p^{(n+1)}(x) \equiv 0$]	M1
			$= \sum_{k=1}^{n} p^{(k)}(x) = q(x) - p(x)$	A1
				[2]
	(ii)	(a)	If $q'(x) = 0$, $q(x) = p(x)$, so the two curves meet at any stationary point.	B1
			But $q(x) > 0$ for large enough $\|x\|$.	M1
			So $q(x)$ has an absolute minimum value	M1
			at which its value is positive, as $q(x) = p(x)$ there.	A1
				[4]
		(b)	$\frac{d}{dx}(e^{-x} q(x)) = e^{-x}(-q(x) + q'(x))$	M1
			$= -e^{-x} p(x) < 0$	A1
			For large enough x, $q(x) > 0$, so $e^{-x} q(x) > 0$,	M1
			but $e^{-x} q(x)$ decreasing, so positive for all x,	A1
			and hence so is $q(x)$.	A1
				[5]
		(c)	$\int_0^\infty p(x+t)e^{-t}\,dt$ $= \left[-p(x+t)e^{-t}\right]_0^\infty + \int_0^\infty p^{(1)}(x+t)e^{-t}\,dt$	M1
			$= (0 - (-p(x))) + \int_0^\infty p^{(1)}(x+t)e^{-t}\,dt$	A1
			So $\int_0^\infty p(x+t)e^{-t}\,dt = p(x) + \int_0^\infty p^{(1)}(x+t)e^{-t}\,dt$ $= p(x) + p^{(1)}(x) + \int_0^\infty p^{(2)}(x+t)e^{-t}\,dt$	M1
			$= p(x) + p^{(1)}(x) + \ldots + p^{(n)}(x) + \int_0^\infty p^{(n+1)}(x+t)e^{-t}\,dt$	M1
			but $p^{(n+1)}(x) \equiv 0$, so $\int_0^\infty p(x+t)e^{-t}\,dt = p(x) + p^{(1)}(x) + \cdots + p^{(n)}(x) = q(x)$.	A1
			but $p(x+t)$, $e^{-t} > 0$ for all $t \geqslant 0$, so $q(x) > 0$.	E1
				[6]

Question			Answer	Mark
4	(i)		$\left(x-\sqrt{2}\right)^2 = 3 \Rightarrow x^2 - 2\sqrt{2}x - 1 = 0$	M1
			so $\left(x^2 - 2\sqrt{2}x - 1\right)\left(x^2 + 2\sqrt{2}x - 1\right) = 0$	M1
			that is, $x^4 - 10x^2 + 1 = 0$	A1
			but $\sqrt{2}+\sqrt{3}$ a root of $\left(x-\sqrt{2}\right)^2 = 3$ so a root of $x^4 - 10x^2 + 1 = 0$.	A1
				[4]
	(ii)		$\left(\sqrt{3}+\sqrt{5}\right)^2 = 8 + 2\sqrt{15}$	B1
			$\sqrt{2}+\sqrt{3}+\sqrt{5}$ a root of $\left(x-\sqrt{2}\right)^2 - \left(8+2\sqrt{15}\right)$	M1
			that is, of $x^2 - 2\sqrt{2}x - 6 - 2\sqrt{15}$	A1
			so of $\left(x^2 - 2\sqrt{2}x - 6 - 2\sqrt{15}\right)\left(x^2 + 2\sqrt{2}x - 6 + 2\sqrt{15}\right)$	M1
			= $x^4 - 20x^2 - 8\sqrt{30}x - 24$	A1
			so of $\left(x^4 - 20x^2 - 8\sqrt{30}x - 24\right)\left(x^4 - 20x^2 + 8\sqrt{30}x - 24\right)$	M1
			= $x^8 - 40x^6 + 352x^4 - 960x^2 + 576$	A1
				[7]
Alternative				
	(ii)		The roots will be the eight numbers of the form $\pm\sqrt{2}\pm\sqrt{3}\pm\sqrt{5}$	B1
			Which can be paired as $\pm\left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right), \pm\left(\sqrt{2}-\sqrt{3}-\sqrt{5}\right),$ $\pm\left(-\sqrt{2}+\sqrt{3}-\sqrt{5}\right), \pm\left(-\sqrt{2}-\sqrt{3}+\sqrt{5}\right)$	
			So the polynomial is a quartic in x^2 with roots $\alpha = \left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)^2, \beta = \left(\sqrt{2}-\sqrt{3}-\sqrt{5}\right)^2,$ $\gamma = \left(-\sqrt{2}+\sqrt{3}-\sqrt{5}\right)^2, \delta = \left(-\sqrt{2}-\sqrt{3}+\sqrt{5}\right)^2$	
			$\alpha+\beta+\gamma+\delta = 4(2+3+5) = 40$	M1
			$\alpha\beta\gamma\delta = \left(2-\left(\sqrt{3}+\sqrt{5}\right)^2\right)^2\left(2-\left(\sqrt{3}-\sqrt{5}\right)^2\right)^2$ $= \left(\left(-6-2\sqrt{15}\right)\left(-6+2\sqrt{15}\right)\right)^2 = (36-60)^2 = 576$	A1
			$\alpha^2 = \left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)^4$ $= 4+9+25+4\left(5\sqrt{6}+7\sqrt{10}+8\sqrt{15}\right)$ $+6(6+10+15)$ $= 224 + 4\left(5\sqrt{6}+7\sqrt{10}+8\sqrt{15}\right)$	
			So $\alpha^2+\beta^2+\gamma^2+\delta^2 = 4\times 224 = 896$	
			$2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) =$ $(\alpha+\beta+\gamma+\delta)^2 - (\alpha^2+\beta^2+\gamma^2+\delta^2)$	M1
			$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{40^2-896}{2} = 352$	A1
			$\alpha^3+\beta^3+\gamma^3+\delta^3 =$ $4(8+27+125+15(12+20+18+45+50+75)+90(30))$ $= 24640$	

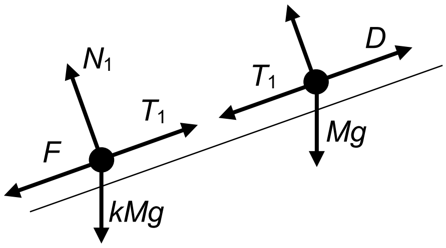
			$(\alpha+\beta+\gamma+\delta)^3 = (\alpha^3+\beta^3+\gamma^3+\delta^3)$ $+3(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)(\alpha+\beta+\gamma+\delta)$ $-3(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)$	M1
			$3(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta) = 24640 + 3(352)(40) - 40^3$	
			$\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta = \frac{2880}{3} = 960$	
			Therefore the polynomial is $x^8 - 40x^6 + 352x^4 - 960x^2 + 576$	A1
				[7]
	(iii)		$a+\sqrt{2}$, $b+\sqrt{2}$, $c+\sqrt{2}$ are roots of $\left(x-\sqrt{2}\right)^3 - 3\left(x-\sqrt{2}\right) + 1$	M1
			so of $x^3 - 3\sqrt{2}x^2 + 3x + \sqrt{2} + 1$	A1
			so of $\left(x^3 - 3\sqrt{2}x^2 + 3x + \sqrt{2} + 1\right)\left(x^3 + 3\sqrt{2}x^2 + 3x - \sqrt{2} + 1\right)$	M1
			$= x^6 - 12x^4 + 2x^3 + 21x^2 + 6x - 1$	A1
				[4]
	(iv)		$\left(\sqrt[3]{2}+\sqrt[3]{3}\right)^3 = 5 + 3\sqrt[3]{12} + 3\sqrt[3]{18}$	M1
			$= 5 + 3\sqrt[3]{6}\left(\sqrt[3]{2}+\sqrt[3]{3}\right)$	A1
			so $\sqrt[3]{2}+\sqrt[3]{3}$ satisfies $x^3 - 5 = 3\sqrt[3]{6}x$	M1
			so satisfies $(x^3-5)^3 = 162x^3$	M1
			or $x^9 - 15x^6 - 87x^3 - 125 = 0$	A1
				[5]

Question			Answer	Mark
5	(i)	(a)	$x_{n+1} - 1 = \frac{1}{x_n+1}$; $x_0 \geqslant 1$	M1
			so if $x_n \geqslant 1$, $x_{n+1} \geqslant 1$	A1
				[2]
		(b)	$x_{n+1}^2 - 2 = \frac{(x_n+2)^2 - 2(x_n+1)^2}{(x_n+1)^2} = -\frac{x_n^2 - 2}{(x_n+1)^2}$	M1
			so $x_{n+1}^2 - 2$ and $x_n^2 - 2$ have opposite sign, as $(x_n+1)^2 \geqslant 2^2 > 0$	A1
			$\lvert x_{n+1}^2 - 2\rvert \leqslant \frac{1}{4}\lvert x_n^2 - 2\rvert$, as $(x_n+1)^2 \geqslant 2^2$	A1
				[3]
		(c)	10 is even, so $x_{10}^2 - 2$ and $x_0^2 - 2$ have the same sign, which is negative, so $x_{10}^2 < 2$	B1
			and $\lvert x_{10}^2 - 2\rvert \leqslant \frac{1}{4^{10}}\lvert x_0^2 - 2\rvert = \frac{1}{4^{10}}$	M1
			$< 10^{-6}$, as $2^{10} > 10^3$	A1
			so $10^{-6} \geqslant 2 - x_{10}^2$ giving stated result	A1
				[4]
	(ii)	(a)	$y_{n+1} - \sqrt{2} = \frac{y_n^2 + 2 - 2\sqrt{2}y_n}{2y_n} = \frac{\left(y_n - \sqrt{2}\right)^2}{2y_n}$	B1
			so $y_0 \geqslant 1$ and $y_{n+1} \geqslant \sqrt{2} \geqslant 1$ for $n \geqslant 0$	B1
				[2]
		(b)	$y_1 - \sqrt{2} = \frac{(1-\sqrt{2})^2}{2} = 2\left(\frac{\sqrt{2}-1}{2}\right)^2$, so result holds for $n = 1$.	B1
			also $y_{n+1} - \sqrt{2} = \frac{(\lvert y_n - \sqrt{2}\rvert)^2}{2y_n} \leqslant \frac{2}{y_n}\left(\frac{\sqrt{2}-1}{2}\right)^{2^{n+1}}$	M1
			$\leqslant 2\left(\frac{\sqrt{2}-1}{2}\right)^{2^{n+1}}$, as $y_n \geqslant \sqrt{2}$ for $n \geqslant 1$	A1
			appropriate induction structure	A1
				[4]
		(c)	$y_{10} \geqslant \sqrt{2}$	B1
			$y_{10} - \sqrt{2} \leqslant 2\left(\frac{\sqrt{2}-1}{2}\right)^{2^{10}}$	M1
			but $\left(\frac{\sqrt{2}-1}{2}\right)^5 \leqslant \left(\frac{1}{4}\right)^5 \leqslant 10^{-3}$ $\left(\frac{\sqrt{2}-1}{2}\right)^5 \leqslant \left(\frac{1}{4}\right)^5 \leqslant 10^{-3}$	M1
			$y_{10} - \sqrt{2} \leqslant 2(10^{-3})^{204}$	A1
			$= 2 \times 10^{-612} < 10^{-600}$	A1
				[5]

Question			Answer	Mark
6			Induction structure	**M1**
			Base case	**B1**
			$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix} = \begin{pmatrix}F_{n+1}+F_n & F_n+F_{n-1}\\ F_{n+1} & F_n\end{pmatrix}$ or $\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}F_{n+1}+F_n & F_{n+1}\\ F_n+F_{n-1} & F_n\end{pmatrix}$	**A1**
			Use of definition (of F_n) and conclusion (of induction)	**A1**
				[4]
	(i)		Use of $\det(Q^n) = (\det Q)^n$	**M1**
			clearly shown	**A1**
				[2]
	(ii)		Use of (1,2) entry in $Q^{m+n} = Q^mQ^n$	**M1**
			clearly shown	**A1**
				[2]
	(iii)		$Q^2 = Q + I$	**B1**
				[1]
		(a)	Use of $Q^{2n} = (Q + I)^n$	**M1**
			and Binomial expansion	**M1**
			clearly shown	**A1**
				[3]
		(b)	Derivation of $Q^3 = Q(Q + I) = 2Q + I$ *(give the mark for any one of these)*	**B1**
			Use of $Q^{3n} = (2Q + I)^n$ and Binomial expansion	**M1**
			clearly shown	**A1**
			Use of $Q^{3n} = Q^n(Q + I)^n$ and Binomial expansion	**M1**
			clearly shown	**A1**
				[5]
		(c)	Use of $I=Q^n(Q-I)^n$ (or (-Q)^n (I-Q)^n)	**M1**
			Use of binomial expansion	**M1**
			clearly shown	**A1**
				[3]

Question			Answer	Mark
7	(i)		$\lvert zw\rvert^2 = \lvert(ac-bd)+i(ad+bc)\rvert^2$ $= (ac-bd)^2 + (ad+bc)^2$ $= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$	**M1**
			$\lvert z\rvert^2\lvert w\rvert^2 = (a^2+b^2)(c^2+d^2)$ $= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$	**M1**
			Therefore $\lvert zw\rvert^2 = \lvert z\rvert^2\lvert w\rvert^2$	**A1**
				[3]
	(ii)		$\lvert 2+i\rvert = \sqrt{5}$ and $\lvert 10+11i\rvert = \sqrt{221}$	**B1**
			so $9^2 + 32^2 = (2^2 + 1^2)(10^2 + 11^2) = 5 \times 221$	**B1**
				[2]
	(iii)		$8045 = 5 \times 1609$	**M1**
			$= (2^2 + 1^2)(40^2 + 3^2)$	**M1**
			so $\lvert(2 + i)(40 + 3i)\rvert^2 = 77^2 + 46^2 = 8045$ (also $34^2 + 83^2$)	**A1**
				[3]
	(iv)		$612^2 + 1206^2 = 6^2 \times 50805$	**B1**
				[1]
	(v)		$1002082 = 1001^2 + 9^2$	**B1**
			so one pair is $5005^2 + 45^2$	**A1**
			but $25 = 3^2 + 4^2$ and $(3 + 4i)(1001 + 9i)$ $= 2967 + 4031i$	**M1**
			so a second pair is $2967^2 + 4031^2$	**A1**
			also, $(4 + 3i)(1001 + 9i) = 3977 + 3039i$	**M1**
			so a third pair is $3977^2 + 3039^2$	**A1**
				[6]
	(vi)		require $(10^2 + 3^2)(c^2 + d^2) = (1001^2 + 6^2)$	**M1**
			implies simultaneous equations for *c* and *d*	**M1**
			$10c - 3d = 1001$, $10d + 3c = 6$ or $3c - 10d = 1001$, $3d + 10c = 6$	**A1**
			giving $c = 92$, $d = -27$ (from the first set)	**A1**
			so $9193 = 92^2 + 27^2$ (also $38^2 + 211^2$)	**A1**
				[5]

Question			Answer	Mark
8	(i)		Let vertices be numbered 1 to 4 and edges be e_{ij}, where $i < j$. Then perimeters equal is $\lvert e_{12}\rvert + \lvert e_{23}\rvert + \lvert e_{13}\rvert = \lvert e_{12}\rvert + \lvert e_{24}\rvert + \lvert e_{14}\rvert$ $= \lvert e_{13}\rvert + \lvert e_{34}\rvert + \lvert e_{14}\rvert$ $= \lvert e_{24}\rvert + \lvert e_{34}\rvert + \lvert e_{23}\rvert$	M1
			which implies $\lvert e_{12}\rvert + \lvert e_{23}\rvert + \lvert e_{13}\rvert + \lvert e_{12}\rvert + \lvert e_{24}\rvert + \lvert e_{14}\rvert$ $= \lvert e_{13}\rvert + \lvert e_{34}\rvert + \lvert e_{14}\rvert + \lvert e_{24}\rvert + \lvert e_{34}\rvert + \lvert e_{23}\rvert$ so $2\lvert e_{12}\rvert + (\lvert e_{23}\rvert + \lvert e_{13}\rvert + \lvert e_{24}\rvert + \lvert e_{14}\rvert)$ $= 2\lvert e_{34}\rvert + (\lvert e_{13}\rvert + \lvert e_{14}\rvert + \lvert e_{24}\rvert + \lvert e_{23}\rvert)$ and so $\lvert e_{12}\rvert = \lvert e_{34}\rvert$	A1
			and by permutations of this argument, all pairs of opposite sides are equal.	E1
			if $\lvert e_{12}\rvert = \lvert e_{34}\rvert$, $\lvert e_{13}\rvert = \lvert e_{24}\rvert$, $\lvert e_{14}\rvert = \lvert e_{23}\rvert$, then all perimeters are trivially equal	B1
				[4]
	(ii)		$\lvert\boldsymbol{a}\rvert^2 = \lvert\boldsymbol{b} - \boldsymbol{c}\rvert^2$	M1
			$= \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 2\boldsymbol{b}.\boldsymbol{c}$	A1
			From the equivalent results to **(ii)** using the other pairs of opposite sides	M1
			$\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c} = \frac{1}{2}(\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 - \lvert\boldsymbol{c}\rvert^2) + \frac{1}{2}(\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{b}\rvert^2) = \lvert\boldsymbol{a}\rvert^2$	A1
				[4]
	(iii)		$16\lvert\boldsymbol{a} - \boldsymbol{g}\rvert^2 = \lvert 3\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}\rvert^2$	M1
			$= 9\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 6\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) + 2\boldsymbol{b}.\boldsymbol{c}$	
			$= 9\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 6\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{a}\rvert^2$ using previous results	M1
			$= 2(\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2)$	A1
			but this is symmetric in **a**, **b**, **c** so **g** equidistant from A, B and C.	A1
			$16\lvert\boldsymbol{g}\rvert^2 = \lvert\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}\rvert^2$ $= \lvert\boldsymbol{a}\rvert^2 + 2\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) + \lvert\boldsymbol{b} + \boldsymbol{c}\rvert^2$ $= \lvert\boldsymbol{a}\rvert^2 + 2\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 + 2\boldsymbol{b}.\boldsymbol{c}$ $= 3\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{a}\rvert^2$ $= 2(\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2)$ So G equidistant from O also.	B1
				[5]
	(iv)		$\lvert\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}\rvert^2 = \lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 2\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) + 2\boldsymbol{b}.\boldsymbol{c}$	M1
			$= \lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 2\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{a}\rvert^2$	M1
			$= 2(\lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{a}\rvert^2)$	A1
			which must be non-negative, so cos(BAC) ⩾ 0	M1
			and symmetry implies no angle obtuse	A1
			If e.g. BAC was a right angle, would have $\lvert\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}\rvert^2 = 0$, so **a** = **b** + **c**	M1
			so O, A, B, C all in one plane, so not a tetrahedron.	A1
				[7]

Question			Answer	Mark
9				B1
			$D - T_1 - Mg \sin \alpha = Ma_1$ **1** $T_1 - F - kMg \sin \alpha = kMa_1$ **2** $[D - \mu kMg\cos \alpha - (k+1)Mg\sin \alpha = (k+1)Ma_1]$	B1
			$N_1 - kMg \cos \alpha = 0$, so $F = \mu kMg \cos \alpha$ **3**	B1
			$D - T_3 + Mg \sin \alpha = Ma_3$ **4** $T_3 - \mu kMg \cos \alpha + kMg \sin \alpha = kMa_3$ **5** $[D - \mu kMg\cos \alpha + (k+1)Mg\sin \alpha = (k+1)Ma_3]$	B1
			$D - T_2 = Ma_2$ **6** $T_2 - \mu kMg = kMa_2$ **7** $[D - \mu kMg = (k+1)Ma_2]$	B1
				[5]
	(i)		**2** – k**1** gives $(1 + k)T_1 = \mu kMg \cos \alpha + kD$	M1
			and **5** – k**4** gives $(1 + k)T_3 = \mu kMg \cos \alpha + kD$ so $T_1 = T_3$.	A1
			1 + **4** – 2**6** gives $-T_1 - T_3 + 2T_2 = Ma_3 + Ma_1 - 2Ma_2$	M1
			which is the given result, using $T_1 = T_3$	A1
				[4]
	(ii)	(a)	**7** – k**6** gives $(1 + k)T_2 = \mu kMg + kD$	M1
			so $T_2 > T_1$, as $\cos \alpha < 1$.	A1
			hence $a_1 + a_3 > 2a_2$,which is the middle inequality	A1
			also, $kMa_1 = T_1 - \mu kMg \cos \alpha - kMg \sin \alpha$ and $kMa_3 = T_3 - \mu kMg \cos \alpha + kMg \sin \alpha$	M1
			so $a_3 > a_1$, which also implies $a_3 > \frac{1}{2}(a_1 + a_3) > a_1$,	A1
				[5]
		(b)	(**7** + **6**) – (**2** + **1**) gives	M1
			$(1+k)(a_2 - a_1) = g\big((1+k)\sin\alpha - k\mu(1-\cos\alpha)\big)$	A1
			$= 2g\sin\left(\frac{1}{2}\alpha\right)\left((1+k)\cos\frac{1}{2}\alpha - k\mu\sin\frac{1}{2}\alpha\right)$	M1
			but $1 + k > \mu k$, as $\mu < 1$, and	M1
			$\cos\frac{1}{2}\alpha > \sin\frac{1}{2}\alpha$, as $\frac{1}{2}\alpha < 45°$	M1
			so $a_2 > a_1$, as required.	A1

				[6]

Question			**Answer**	**Mark**
10	**(i)**		$z = ut\sin\alpha - \frac{1}{2}gt^2$ and $x = ut\cos\alpha$	**M1**
			require $ut\sin\alpha - \frac{1}{2}gt^2 = A - B(ut\cos\alpha)^2$ to have a double root	**M1**
			so $\left(Bu^2\cos^2\alpha - \frac{1}{2}g\right)t^2 + (u\sin\alpha)t - A = 0$ has zero discriminant	**A1**
			so $u^2\sin^2\alpha + 4A\left(Bu^2\cos^2\alpha - \frac{1}{2}g\right) = 0$	**A1**
			using $\sin^2\alpha = 1 - \cos^2\alpha$, gives $u^2 - u^2\cos^2\alpha + 4ABu^2\cos^2\alpha - 2Ag = 0$	**A1**
			If this is true for all α, both the left-hand side, and the coefficient of $\cos^2\alpha$ on the right-hand side must be zero	**M1**
			So $A = \frac{u^2}{2g}$ and $B = \frac{g}{2u^2}$	**A1**
				[7]
	(ii)		The set of points inside and on S are vulnerable	**M1**
			because if a point outside S was on a trajectory, it would have to cross S, which contradicts the fact that it touches to the surface.	**A1**
				[2]
	(iii)		Particles in the *x*-*z* plane have, eliminating α,	**M1**
			$$\left(z + \frac{1}{2}gt^2\right)^2 + x^2 = u^2t^2$$	**A1**
			so, rotating this circle about the *z*-axis, all the particles lie on a sphere	**M1**
			of radius *ut* and centre $\left(0,\ 0,\ -\frac{1}{2}gt^2\right)$	**A1**
				[4]
	(iv)		for *Q*, $z = \frac{u^2}{2g} - \frac{1}{2}gt^2$ and $x = ut$	**M1**
			so $z = \frac{u^2}{2g} - \frac{gx^2}{2u^2}$, which is the equation of *E*	**A1**
				[2]
	(v)		Using **(i)**, P_α meets *E* at time *t* given by	**M1**
			$$g^2\sin^2\alpha\, t^2 - 2gu\sin\alpha\, t + u^2 = 0$$	**A1**
			so at $t = \frac{u}{g\sin\alpha}$	**A1**
			so require $u(t - T) = u\cos\alpha\, t$	**M1**
			that is $T = \frac{u}{g}\frac{1-\cos\alpha}{\sin\alpha}$	**A1**
				[5]

Question			Answer	Mark
11	(i)		E[Y] = $\sum_{r=1}^{n} x_r(pa_r + qb_r) = p\sum_{r=1}^{n} x_r a_r + q\sum_{r=1}^{n} x_r b_r = p\mu_1 + q\mu_2$	B1
			Var[Y] = $\sum_{r=1}^{n} x^2{}_r(pa_r + qb_r) - E[Y]^2$ = $p(\sigma_1{}^2 + \mu_1{}^2) + q(\sigma_2{}^2 + \mu_2{}^2) - (p\mu_1 + q\mu_2)^2$	M1
			$= p\sigma_1{}^2 + q\sigma_2{}^2 + (p - p^2)\mu_1{}^2 + 2pq\mu_1\mu_2 + (q - q^2)\mu_2{}^2$	M1
			giving the required result	A1
				[4]
	(ii)		B = 1 with probability $\frac{1}{2}\cdot\frac{1}{6} + \frac{1}{2}\cdot\frac{5}{6} = \frac{1}{2}$ and Z_1 counts the number of times out of n that B = 1	B1
			mean of Z_1 = $\frac{1}{2}n$ and variance of Z_1 = $\frac{1}{4}n$	B1
			so $P\left(\frac{1}{2}n - \frac{1}{20}n \leqslant Z_1 \leqslant \frac{1}{2}n + \frac{1}{20}n\right)$	M1
			$\approx P\left(-\frac{\frac{1}{20}n}{\frac{1}{2}\sqrt{n}} \leqslant S \leqslant \frac{\frac{1}{20}n}{\frac{1}{2}\sqrt{n}}\right)$ where S is standard Normal	A1
			but $\sqrt{n}$ gets large as n gets large/n grows faster than $\sqrt{n}$ as n gets large	M1
			and hence the probability tends to $P(-\infty < S < \infty) = 1$ 1 as $n \to \infty$	A1
				[6]
	(iii)		mean of Z_2 = $\frac{1}{2}\cdot\frac{1}{6}n + \frac{1}{2}\cdot\frac{5}{6}n = \frac{1}{2}n$	B1
			and variance of Z_2 = $\frac{1}{2}\cdot\frac{5}{36}n + \frac{1}{2}\cdot\frac{5}{36}n + \frac{1}{4}\left(\frac{1}{6}n - \frac{5}{6}n\right)^2$ = $\frac{5}{36}n + \frac{1}{9}n^2$	B1
			A Normal distribution with this mean and variance will not be a good approximation to the distribution of Z_2 because Z_2 is bimodal: it is likely to take values close to $\frac{1}{6}n$ or $\frac{5}{6}n$, not near $\frac{1}{2}n$	B1
			Z_2 is within 10% of its mean only if the coin shows heads and a surprisingly large number of sixes appear, or the coin shows tails, and surprisingly few sixes appear.	M1
			In the first case, the probability that a surprisingly large number of heads appears is less than	M1
			$P\left(B_1 \geqslant \frac{1}{2}n - \frac{1}{20}n\right)$ where B_1 is a Binomial variable with mean $\frac{1}{6}n$ and variance $\frac{5}{36}n$.	A1
			which, by Normal approximation, $\approx P\left(S \geqslant \frac{\frac{1}{2}n - \frac{1}{20}n - \frac{1}{6}n}{\frac{1}{6}\sqrt{5n}}\right)$	M1
			= $P\left(S \geqslant \frac{17\sqrt{n}}{10\sqrt{5}}\right)$	A1

			This is also greater than the probability of surprisingly few heads in the second case so also the probability that Z_2 is within 10% of its mean	**M1**
			and tends to 0 as $n \to \infty$	**A1**
				[10]

Question			Answer	Mark
12	(i)		$P(Y \leqslant t) = P(X_i \leqslant t$ for all $i = 1, \ldots, n)$ $= \prod_{i=1}^{n} P(X_i \leqslant t) = [P(X_1 \leqslant t)]^n$	E1
			$= \left(\int_0^t \frac{1}{2}\sin x\,dx\right)^n = \frac{1}{2^n}(1-\cos t)^n$	M1
			so $f_Y(t) = \frac{n\sin t}{2^n}(1-\cos t)^{n-1}$	A1
				[3]
	(ii)		$\frac{1}{2^n}(1-\cos m(n))^n = \frac{1}{2}$	M1
			so $m(n) = \arccos\left(1 - 2^{\frac{n-1}{n}}\right)$	A1
			which tends to π as $n \to \infty$.	A1
				[3]
	(iii)		$\mu(n) = \int_0^{\pi} x\frac{n}{2^n}\sin x(1-\cos x)^{n-1}\,dx$ $= \left[x\frac{1}{2^n}(1-\cos x)^n\right]_0^{\pi} - \int_0^{\pi}\frac{1}{2^n}(1-\cos x)^n\,dx$	M1
			$= \left(\pi\frac{1}{2^n}2^n - 0\right) - \int_0^{\pi}\frac{1}{2^n}(1-\cos x)^n\,dx$ as required	A1
				[2]
		(a)	As $\mu(n) = \pi - \int_0^{\pi}\left(\frac{1-\cos x}{2}\right)^n dx$, the integrand decreases with n throughout $(0, \pi)$	M1
			and so $\mu(n)$ increases with n	A1
				[2]
		(b)	$\mu(2) = \pi - \int_0^{\pi}\frac{1}{4}(1 - 2\cos x + \cos^2 x)\,dx$ $= \pi - \int_0^{\pi}\frac{1}{4}\left(1 - 2\cos x + \frac{1}{2}(1+\cos 2x)\right)dx$	M1
			$= \pi - \left[\frac{3}{8}x - \frac{1}{2}\sin x + \frac{1}{16}\sin 2x\right]_0^{\pi}$	M1
			$= \frac{5}{8}\pi$	A1
			so $\cos^2(\mu(2)) = \frac{1}{2}\left(1 + \cos\frac{5}{4}\pi\right) = \frac{1}{4}(2 - \sqrt{2})$	M1
			but $\cos^2(m(2)) = (1 - \sqrt{2})^2 = 3 - 2\sqrt{2}$	M1
			which is greater, as $(3 - 2\sqrt{2}) - \frac{1}{4}(2 - \sqrt{2}) = \frac{1}{4}(10 - 7\sqrt{2})$	M1
			$= \frac{1}{2(10 + 7\sqrt{2})} > 0$	A1
			but both values are between $\frac{1}{2}\pi$ and π,	M1
			so both cosines are negative and hence $\cos^2(m(2)) > \cos^2(\mu(2)) \Rightarrow 0 > \cos(\mu(2)) > \cos(m(2))$	M1
			so $m(2) > \mu(2)$	A1
				[10]