Cambridge Assessment
Admissions Testing

Sixth Term Examination Papers **9470**

MATHEMATICS 2 Morning

Monday 14 June 2021 Time: 3 hours

1261392748

Additional Material: Answer Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

INFORMATION FOR CANDIDATES

There are 12 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You must shade the appropriate Question Answered circle on every page of the answer booklet that you write on. Failure to do so might mean that some of your answers are not marked.

There is NO Mathematical Formulae Booklet.

Calculators are not permitted.

Wait to be told you may begin before turning this page.

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Section A: Pure Mathematics

1 Prove, from the identities for $\cos(A \pm B)$, that $\cos a \cos 3a \equiv \frac{1}{2}(\cos 4a + \cos 2a)$.

Find a similar identity for $\sin a \cos 3a$.

(i) Solve the equation

$$4 \cos x \cos 2x \cos 3x = 1$$

for $0 \leqslant x \leqslant \pi$.

(ii) Prove that if

$$\tan x = \tan 2x \tan 3x \tan 4x \qquad (\star)$$

then $\cos 6x = \frac{1}{2}$ or $\sin 4x = 0$.

Hence determine the solutions of equation $(\star)$ with $0 \leqslant x \leqslant \pi$.

2 In this question, the numbers a, b and c may be complex.

(i) Let p, q and r be real numbers. Given that there are numbers a and b such that

$$a + b = p\,, \quad a^2 + b^2 = q \text{ and } a^3 + b^3 = r\,, \qquad (*)$$

show that $3pq - p^3 = 2r$.

(ii) Conversely, you are given that the real numbers p, q and r satisfy $3pq - p^3 = 2r$. By considering the equation $2x^2 - 2px + (p^2 - q) = 0$, show that there exist numbers a and b such that the three equations $(*)$ hold.

(iii) Let s, t, u and v be real numbers. Given that there are distinct numbers a, b and c such that

$$a + b + c = s\,, \quad a^2 + b^2 + c^2 = t\,, \quad a^3 + b^3 + c^3 = u \text{ and } abc = v\,,$$

show, using part **(i)**, that c is a root of the equation

$$6x^3 - 6sx^2 + 3(s^2 - t)x + 3st - s^3 - 2u = 0$$

and write down the other two roots.

Deduce that $s^3 - 3st + 2u = 6v$.

(iv) Find numbers a, b and c such that

$$a + b + c = 3\,, \quad a^2 + b^2 + c^2 = 1\,, \quad a^3 + b^3 + c^3 = -3 \text{ and } abc = 2\,, \qquad (**)$$

and verify that your solution satisfies the four equations $(**)$.

3 In this question, x, y and z are real numbers.

Let $\lfloor x \rfloor$ denote the largest integer that satisfies $\lfloor x \rfloor \leqslant x$ and let $\{x\}$ denote the fractional part of x, so that $x = \lfloor x \rfloor + \{x\}$ and $0 \leqslant \{x\} < 1$. For example, if $x = 4.2$, then $\lfloor x \rfloor = 4$ and $\{x\} = 0.2$ and if $x = -4.2$, then $\lfloor x \rfloor = -5$ and $\{x\} = 0.8$.

(i) Solve the simultaneous equations

$$\begin{aligned} \lfloor x \rfloor + \{y\} &= 4.9\,, \\ \{x\} + \lfloor y \rfloor &= -1.4\,. \end{aligned}$$

(ii) Given that x, y and z satisfy the simultaneous equations

$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 3.9\,, \\ \{x\} + \ y \ + \lfloor z \rfloor &= 5.3\,, \\ \lfloor x \rfloor + \{y\} \ + z \ &= 5\,, \end{aligned}$$

show that $\{y\} + \lfloor z \rfloor = 3.2$ and solve the equations.

(iii) Solve the simultaneous equations

$$\begin{aligned} x + 2\lfloor y \rfloor + \{z\} &= 3.9\,, \\ \{x\} + \ 2y \ + \lfloor z \rfloor &= 5.3\,, \\ \lfloor x \rfloor + 2\{y\} \ + z \ &= 5\,. \end{aligned}$$

4 **(i)** Sketch the curve $y = x\mathrm{e}^x$, giving the coordinates of any stationary points.

(ii) The function f is defined by $\mathrm{f}(x) = x\mathrm{e}^x$ for $x \geqslant a$, where a is the minimum possible value such that f has an inverse function. What is the value of a?

Let g be the inverse of f. Sketch the curve $y = \mathrm{g}(x)$.

(iii) For each of the following equations, find a real root in terms of a value of the function g, or demonstrate that the equation has no real root. If the equation has two real roots, determine whether the root you have found is greater than or less than the other root.

(a) $\mathrm{e}^{-x} = 5x$ **(b)** $2x \ln x + 1 = 0$ **(c)** $3x \ln x + 1 = 0$ **(d)** $x = 3 \ln x$

(iv) Given that the equation $x^x = 10$ has a unique positive root, find this root in terms of a value of the function g.

5 **(i)** Use the substitution $y = (x - a)u$, where u is a function of x, to solve the differential equation

$$(x-a)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x\,,$$

where a is a constant.

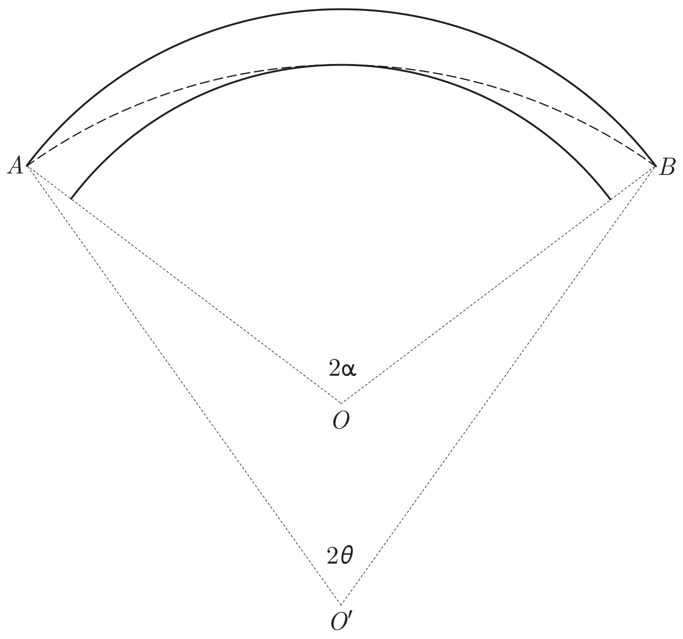
(ii) The curve C with equation $y = \mathrm{f}(x)$ has the property that, for all values of t except $t = 1$, the tangent at the point $\big(t, \mathrm{f}(t)\big)$ passes through the point $(1, t)$.

(a) Given that $\mathrm{f}(0) = 0$, find $\mathrm{f}(x)$ for $x < 1$.

Sketch C for $x < 1$. You should find the co-ordinates of any stationary points and consider the gradient of C as $x \to 1$. You may assume that $z \ln |z| \to 0$ as $z \to 0$.

(b) Given that $\mathrm{f}(2) = 2$, sketch C for $x > 1$, giving the co-ordinates of any stationary points.

6 A plane circular road is bounded by two concentric circles with centres at point O. The inner circle has radius R and the outer circle has radius $R + w$. The points A and B lie on the outer circle, as shown in the diagram, with $\angle AOB = 2\alpha$, $\frac{1}{3}\pi \leqslant \alpha \leqslant \frac{1}{2}\pi$ and $0 < w < R$.



(i) Show that I cannot cycle from A to B in a straight line, while remaining on the road.

(ii) I take a path from A to B that is an arc of a circle. This circle is tangent to the inner edge of the road, and has radius $R + d$ (where $d > w$) and centre O'.

My path is represented by the dashed arc in the above diagram.

Let $\angle AO'B = 2\theta$.

(a) Use the cosine rule to find d in terms of w, R and $\cos\alpha$.

(b) Find also an expression for $\sin(\alpha - \theta)$ in terms of R, d and $\sin\alpha$.

You are now given that $\dfrac{w}{R}$ is much less than 1.

(iii) Show that $\dfrac{d}{R}$ and $\alpha - \theta$ are also both much less than 1.

(iv) My friend cycles from A to B along the outer edge of the road.

Let my path be shorter than my friend's path by distance S. Show that

$$S = 2(R + d)(\alpha - \theta) + 2\alpha(w - d).$$

Hence show that S is approximately a fraction

$$\left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha(1 - \cos\alpha)}\right)\frac{w}{R}$$

of the length of my friend's path.

7 **(i)** The matrix $\mathbf{R}$ represents an anticlockwise rotation through angle ϕ ($0^\circ \leqslant \phi < 360^\circ$) in two dimensions, and the matrix $\mathbf{R} + \mathbf{I}$ also represents a rotation in two dimensions. Determine the possible values of ϕ and deduce that $\mathbf{R}^3 = \mathbf{I}$.

(ii) Let $\mathbf{S}$ be a real matrix with $\mathbf{S}^3 = \mathbf{I}$, but $\mathbf{S} \neq \mathbf{I}$.

Show that $\det(\mathbf{S}) = 1$.

Given that

$$\mathbf{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

show that $\mathbf{S}^2 = (a+d)\mathbf{S} - \mathbf{I}$.

Hence prove that $a + d = -1$.

(iii) Let $\mathbf{S}$ be a real 2×2 matrix.

Show that if $\mathbf{S}^3 = \mathbf{I}$ and $\mathbf{S} + \mathbf{I}$ represents a rotation, then $\mathbf{S}$ also represents a rotation. What are the possible angles of the rotation represented by $\mathbf{S}$?

8 **(i)** Show that, for $n = 2, 3, 4, \ldots,$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(t^n(1-t)^n\Big) = nt^{n-2}(1-t)^{n-2}\Big[(n-1) - 2(2n-1)t(1-t)\Big].$$

(ii) The sequence $T_0, T_1, \ldots$ is defined by

$$T_n = \int_0^1 \frac{t^n(1-t)^n}{n!}\, \mathrm{e}^t \, \mathrm{d}t\,.$$

Show that, for $n \geqslant 2$,

$$T_n = T_{n-2} - 2(2n-1)T_{n-1}\,.$$

(iii) Evaluate T_0 and T_1 and deduce that, for $n \geqslant 0$, T_n can be written in the form

$$T_n = a_n + b_n \mathrm{e}\,,$$

where a_n and b_n are integers (which you should not attempt to evaluate).

(iv) Show that $0 < T_n < \dfrac{\mathrm{e}}{n!}$ for $n \geqslant 0$. Given that b_n is non-zero for all n, deduce that $\dfrac{-a_n}{b_n}$ tends to e as n tends to infinity.

Section B: Mechanics

9 Two particles, of masses m_1 and m_2 where $m_1 > m_2$, are attached to the ends of a light, inextensible string. A particle of mass M is fixed to a point P on the string. The string passes over two small, smooth pulleys at Q and R, where QR is horizontal, so that the particle of mass m_1 hangs vertically below Q and the particle of mass m_2 hangs vertically below R. The particle of mass M hangs between the two pulleys with the section of the string PQ making an acute angle of θ_1 with the upward vertical and the section of the string PR making an acute angle of θ_2 with the upward vertical. S is the point on QR vertically above P. The system is in equilibrium.

(i) Using a triangle of forces, or otherwise, show that:

(a)

$$\sqrt{m_1^2 - m_2^2} < M < m_1 + m_2\,;$$

(b) S divides QR in the ratio $r : 1$, where

$$r = \frac{M^2 - m_1^2 + m_2^2}{M^2 - m_2^2 + m_1^2}\,.$$

(ii) You are now given that $M^2 = m_1^2 + m_2^2$.

Show that $\theta_1 + \theta_2 = 90°$ and determine the ratio of QR to SP in terms of the masses only.

10 A train moves westwards on a straight horizontal track with constant acceleration a, where $a > 0$. Axes are chosen as follows: the origin is fixed in the train; the x-axis is in the direction of the track with the positive x-axis pointing to the East; and the positive y-axis points vertically upwards.

A smooth wire is fixed in the train. It lies in the x-y plane and is bent in the shape given by $ky = x^2$, where k is a positive constant. A small bead is threaded onto the wire. Initially, the bead is held at the origin. It is then released.

(i) Explain why the bead cannot remain stationary relative to the train at the origin.

(ii) Show that, in the subsequent motion, the coordinates (x, y) of the bead satisfy

$$\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) = 0$$

and deduce that $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy$ is constant during the motion.

(iii) Find an expression for the maximum vertical displacement, b, of the bead from its initial position in terms of a, k and g.

(iv) Find the value of x for which the speed of the bead relative to the train is greatest and give this maximum speed in terms of a, k and g.

Section C: Probability and Statistics

11 A train has n seats, where $n \geqslant 2$. For a particular journey, all n seats have been sold, and each of the n passengers has been allocated a seat.

The passengers arrive one at a time and are labelled $T_1, \ldots, T_n$ according to the order in which they arrive: T_1 arrives first and T_n arrives last. The seat allocated to T_r ($r = 1, \ldots, n$) is labelled S_r.

Passenger T_1 ignores their allocation and decides to choose a seat at random (each of the n seats being equally likely). However, for each $r \geqslant 2$, passenger T_r sits in S_r if it is available or, if S_r is not available, chooses from the available seats at random.

(i) Let P_n be the probability that, in a train with n seats, T_n sits in S_n. Write down the value of P_2 and find the value of P_3.

(ii) Explain why, for $k = 2, 3, \ldots, n-1$,

$$\mathrm{P}\big(T_n \text{ sits in } S_n \mid T_1 \text{ sits in } S_k\big) = P_{n-k+1}\,,$$

and deduce that, for $n \geqslant 3$,

$$P_n = \frac{1}{n}\left(1 + \sum_{r=2}^{n-1} P_r\right).$$

(iii) Give the value of P_n in its simplest form and prove your result by induction.

(iv) Let Q_n be the probability that, in a train with n seats, T_{n-1} sits in S_{n-1}. Determine Q_n for $n \geqslant 2$.

12 **(i)** A game for two players, A and B, can be won by player A, with probability p_{A}, won by player B, with probability p_{B}, where $0 < p_{\mathrm{A}} + p_{\mathrm{B}} < 1$, or drawn. A match consists of a series of games and is won by the first player to win a game. Show that the probability that A wins the match is

$$\frac{p_{\mathrm{A}}}{p_{\mathrm{A}} + p_{\mathrm{B}}}\,.$$

(ii) A second game for two players, A and B, can be won by player A, with probability p, or won by player B, with probability $q = 1 - p$. A match consists of a series of games and is won by the first player to have won two more games than the other. Show that the match is won after an even number of games, and that the probability that A wins the match is

$$\frac{p^2}{p^2 + q^2}\,.$$

(iii) A third game, for only one player, consists of a series of rounds. The player starts the game with one token, wins the game if they have four tokens at the end of a round and loses the game if they have no tokens at the end of a round. There are two versions of the game. In the cautious version, in each round where the player has any tokens, the player wins one token with probability p and loses one token with probability $q = 1-p$. In the bold version, in each round where the player has any tokens, the player's tokens are doubled in number with probability p and all lost with probability $q = 1 - p$.

In each of the two versions of the game, find the probability that the player wins.

Hence show that the player is more likely to win in the cautious version if $1 > p > \frac{1}{2}$ and more likely to win in the bold version if $0 < p < \frac{1}{2}$.

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