Admissions Testing Service

Sixth Term Examination Papers **9470**

MATHEMATICS 2 Morning

WEDNESDAY 17 JUNE 2015 Time: 3 hours

8493300458

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Please read this page carefully, but do not open this question paper until you are told that you may do so.

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Please wait to be told you may begin before turning this page.

This question paper consists of 8 printed pages and 4 blank pages.

BLANK PAGE

Section A: Pure Mathematics

1 **(i)** By use of calculus, show that $x - \ln(1+x)$ is positive for all positive x. Use this result to show that

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)\,.$$

(ii) By considering $x + \ln(1-x)$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2\,.$$

2 In the triangle ABC, angle $BAC = \alpha$ and angle $CBA = 2\alpha$, where 2α is acute, and $BC = x$. Show that $AB = (3 - 4\sin^2\alpha)x$.

The point D is the midpoint of AB and the point E is the foot of the perpendicular from C to AB. Find an expression for DE in terms of x.

The point F lies on the perpendicular bisector of AB and is a distance x from C. The points F and B lie on the same side of the line through A and C. Show that the line FC trisects the angle ACB.

3 Three rods have lengths a, b and c, where $a < b < c$. The three rods can be made into a triangle (possibly of zero area) if $a + b \geqslant c$.

Let T_n be the number of triangles that can be made with three rods chosen from n rods of lengths 1, 2, 3, … , n (where $n \geqslant 3$). Show that $T_8 - T_7 = 2 + 4 + 6$ and evaluate $T_8 - T_6$. Write down expressions for $T_{2m} - T_{2m-1}$ and $T_{2m} - T_{2m-2}$.

Prove by induction that $T_{2m} = \frac{1}{6}m(m-1)(4m+1)$, and find the corresponding result for an odd number of rods.

4 **(i)** The continuous function f is defined by

$$\tan \mathrm{f}(x) = x \qquad (-\infty < x < \infty)$$

and $\mathrm{f}(0) = \pi$. Sketch the curve $y = \mathrm{f}(x)$.

(ii) The continuous function g is defined by

$$\tan \mathrm{g}(x) = \frac{x}{1+x^2} \qquad (-\infty < x < \infty)$$

and $\mathrm{g}(0) = \pi$. Sketch the curves $y = \dfrac{x}{1+x^2}$ and $y = \mathrm{g}(x)$.

(iii) The continuous function h is defined by $\mathrm{h}(0) = \pi$ and

$$\tan \mathrm{h}(x) = \frac{x}{1-x^2} \qquad (x \neq \pm 1).$$

(The values of $\mathrm{h}(x)$ at $x = \pm 1$ are such that $\mathrm{h}(x)$ is continuous at these points.) Sketch the curves $y = \dfrac{x}{1-x^2}$ and $y = \mathrm{h}(x)$.

5 In this question, the arctan function satisfies $0 \leqslant \arctan x < \frac{1}{2}\pi$ for $x \geqslant 0$.

(i) Let

$$S_n = \sum_{m=1}^{n} \arctan\left(\frac{1}{2m^2}\right),$$

for $n = 1, 2, 3, \ldots$. Prove by induction that

$$\tan S_n = \frac{n}{n+1}.$$

Prove also that

$$S_n = \arctan \frac{n}{n+1}.$$

(ii) In a triangle ABC, the lengths of the sides AB and BC are $4n^2$ and $4n^4-1$, respectively, and the angle at B is a right angle. Let angle $BCA = 2\alpha_n$. Show that

$$\sum_{n=1}^{\infty} \alpha_n = \tfrac{1}{4}\pi.$$

6 **(i)** Show that

$$\sec^2\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right) = \frac{2}{1+\sin x}\,.$$

Hence integrate $\dfrac{1}{1+\sin x}$ with respect to x.

(ii) By means of the substitution $y = \pi - x$, show that

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \mathrm{f}(\sin x)\,\mathrm{d}x,$$

where f is any function for which these integrals exist.

Hence evaluate

$$\int_0^{\pi} \frac{x}{1+\sin x}\,\mathrm{d}x\,.$$

(iii) Evaluate

$$\int_0^{\pi} \frac{2x^3 - 3\pi x^2}{(1+\sin x)^2}\,\mathrm{d}x.$$

7 A circle C is said to be *bisected* by a curve X if X meets C in exactly two points and these points are diametrically opposite each other on C.

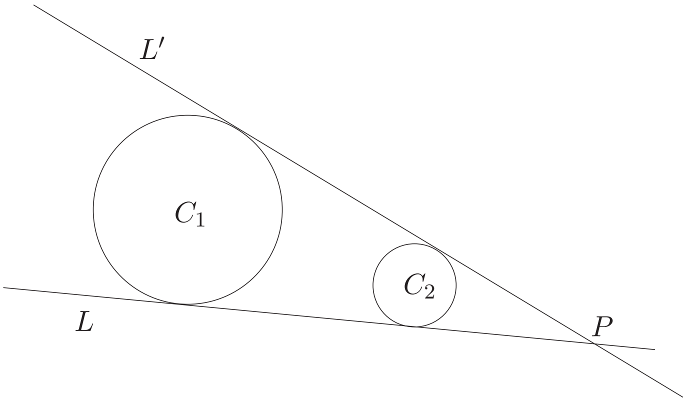
(i) Let C be the circle of radius a in the x-y plane with centre at the origin. Show, by giving its equation, that it is possible to find a circle of given radius r that bisects C provided $r > a$. Show that no circle of radius r bisects C if $r \leqslant a$.

(ii) Let C_1 and C_2 be circles with centres at $(-d, 0)$ and $(d, 0)$ and radii a_1 and a_2, respectively, where $d > a_1$ and $d > a_2$. Let D be a circle of radius r that bisects both C_1 and C_2. Show that the x-coordinate of the centre of D is $\dfrac{a_2^2 - a_1^2}{4d}$.

Obtain an expression in terms of d, r, a_1 and a_2 for the y-coordinate of the centre of D, and deduce that r must satisfy

$$16r^2d^2 \geqslant \left(4d^2 + (a_2 - a_1)^2\right)\left(4d^2 + (a_2 + a_1)^2\right).$$

8



The diagram above shows two non-overlapping circles C_1 and C_2 of different sizes. The lines L and L' are the two common tangents to C_1 and C_2 such that the two circles lie on the same side of each of the tangents. The lines L and L' intersect at the point P which is called the *focus* of C_1 and C_2.

(i) Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the position vectors of the centres of C_1 and C_2, respectively. Show that the position vector of P is

$$\frac{r_1\mathbf{x}_2 - r_2\mathbf{x}_1}{r_1 - r_2},$$

where r_1 and r_2 are the radii of C_1 and C_2, respectively.

(ii) The circle C_3 does not overlap either C_1 or C_2 and its radius, r_3, satisfies $r_1 \neq r_3 \neq r_2$. The focus of C_1 and C_3 is Q, and the focus of C_2 and C_3 is R. Show that P, Q and R lie on the same straight line.

(iii) Find a condition on r_1, r_2 and r_3 for Q to lie half-way between P and R.

Section B: Mechanics

9 An equilateral triangle ABC is made of three light rods each of length a. It is free to rotate in a vertical plane about a horizontal axis through A. Particles of mass $3m$ and $5m$ are attached to B and C respectively. Initially, the system hangs in equilibrium with BC below A.

(i) Show that, initially, the angle θ that BC makes with the horizontal is given by $\sin\theta = \frac{1}{7}$.

(ii) The triangle receives an impulse that imparts a speed v to the particle B. Find the minimum speed v_0 such that the system will perform complete rotations if $v > v_0$.

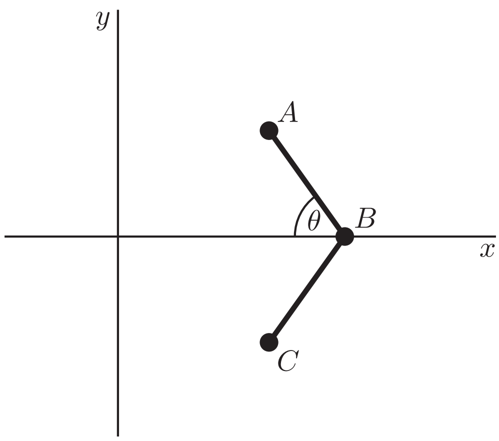
10 A particle of mass m is pulled along the floor of a room in a straight line by a light string which is pulled at constant speed V through a hole in the ceiling. The floor is smooth and horizontal, and the height of the room is h. Find, in terms of V and θ, the speed of the particle when the string makes an angle of θ with the vertical (and the particle is still in contact with the floor). Find also the acceleration, in terms of V, h and θ.

Find the tension in the string and hence show that the particle will leave the floor when

$$\tan^4\theta = \frac{V^2}{gh}\,.$$

11 Three particles, A, B and C, each of mass m, lie on a smooth horizontal table. Particles A and C are attached to the two ends of a light inextensible string of length $2a$ and particle B is attached to the midpoint of the string. Initially, A, B and C are at rest at points $(0, a)$, $(0, 0)$ and $(0, -a)$, respectively.

An impulse is delivered to B, imparting to it a speed u in the positive x direction. The string remains taut throughout the subsequent motion.



(i) At time t, the angle between the x-axis and the string joining A and B is θ, as shown in the diagram, and B is at $(x, 0)$. Write down the coordinates of A in terms of x, a and θ at this time. Given that the velocity of B is $(v, 0)$, show that the velocity of A is $(v + a\dot{\theta}\sin\theta\,,\ a\dot{\theta}\cos\theta)$, where the dot denotes differentiation with respect to time.

(ii) Show that, before A and C first collide,

$$3v + 2a\dot{\theta}\sin\theta = u \qquad \text{and} \qquad \dot{\theta}^2 = \frac{u^2}{a^2(3 - 2\sin^2\theta)}\,.$$

(iii) When A and C collide, the collision is elastic (no energy is lost). At what value of θ does the second collision between particles A and C occur? (You should justify your answer.)

(iv) When $v = 0$, what are the possible values of θ? Is $v = 0$ whenever θ takes these values?

[Turn over

Section C: Probability and Statistics

12 Four players A, B, C and D play a coin-tossing game with a fair coin. Each player chooses a sequence of heads and tails, as follows:

Player A: HHT; Player B: THH; Player C: TTH; Player D: HTT.

The coin is then tossed until one of these sequences occurs, in which case the corresponding player is the winner.

(i) Show that, if only A and B play, then A has a probability of $\frac{1}{4}$ of winning.

(ii) If all four players play together, find the probabilities of each one winning.

(iii) Only B and C play. What is the probability of C winning if the first two tosses are TT?

Let the probabilities of C winning if the first two tosses are HT, TH and HH be p, q and r, respectively. Show that $p = \frac{1}{2} + \frac{1}{2}q$.

Find the probability that C wins.

13 The maximum height X of flood water each year on a certain river is a random variable with probability density function f given by

$$\mathrm{f}(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & \text{for } x \geqslant 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive constant.

It costs ky pounds each year to prepare for flood water of height y or less, where k is a positive constant and $y \geqslant 0$. If $X \leqslant y$ no further costs are incurred but if $X > y$ the additional cost of flood damage is $a(X - y)$ pounds where a is a positive constant.

(i) Let C be the total cost of dealing with the floods in the year. Show that the expectation of C is given by

$$\mathrm{E}(C) = ky + \frac{a}{\lambda}\mathrm{e}^{-\lambda y}\,.$$

How should y be chosen in order to minimise $\mathrm{E}(C)$, in the different cases that arise according to the value of a/k?

(ii) Find the variance of C, and show that the more that is spent on preparing for flood water in advance the smaller this variance.

BLANK PAGE

BLANK PAGE

BLANK PAGE