STEP 2022, P3, Q8 - Solution (5 pages; 17/2/24)

(i) **1st Part**

By De Moivre's theorem,

$(cos\theta + isin\theta)^k = \cos(k\theta) + isin(k\theta)$ (*)

Also, $(cos\theta + isin\theta)^k = cos^k\theta(1 + itan\theta)^k$

$= cos^k\theta(1 + kitan\theta + \binom{k}{2}(itan\theta)^2 + \binom{k}{3}(itan\theta)^3$

$+ \binom{k}{4}(itan\theta)^4 + \binom{k}{5}(itan\theta)^5 + \cdots)$ (**) (for positive integer k)

Then, equating Im. parts of (*) and (**),

$sin(k\theta) = cos^k\theta(ktan\theta - \binom{k}{3}tan^3\theta + \binom{k}{5}tan^5\theta - \cdots)$

$= cos^k\theta tan\theta(k - \binom{k}{3}tan^2\theta + \binom{k}{5}tan^4\theta - \cdots)$

$= sin\theta cos^{k-1}\theta(k - \binom{k}{3}(sec^2\theta - 1) + \binom{k}{5}(sec^2\theta - 1)^2 - \cdots)$,

as required

2nd Part

Equating Re. parts of (*) and (**),

$cos(k\theta) = cos^k\theta(1 - \binom{k}{2}(sec^2\theta - 1) + \binom{k}{4}(sec^2\theta - 1)^2 - \cdots)$

[These expressions for $sin(k\theta)$ and $cos(k\theta)$ are in fact valid for $k = 1$ as well.]

(ii) **1st Part**

Write $sin(k\theta) = sin\theta cos^{k-1}\theta\left(k - \binom{k}{3}(sec^2\theta - 1\right)$

$+\binom{k}{5}(sec^2\theta - 1)^2 - \cdots)$

Now, $\theta = cos^{-1}\left(\frac{1}{a}\right) \Rightarrow cos\theta = \frac{1}{a}$, so that $sin\theta \neq 0$ (otherwise a would be 1).

And $cos^{k-1}\theta = \frac{1}{a^k} \neq 0$.

Also, $sec^2\theta - 1 = a^2 - 1$, which is even, as a is odd.

Then $sin(k\theta) = 0$

$\Rightarrow k - \binom{k}{3}(sec^2\theta - 1) + \binom{k}{5}(sec^2\theta - 1)^2 - \cdots = 0$

and so, as all the terms after the 1st are even, it follows that k must be even.

2nd Part

$sin(k\theta) = 0 \Rightarrow 2\sin\left(\frac{1}{2}k\theta\right)\cos\left(\frac{1}{2}k\theta\right) = 0$

We are told that $\sin(m\theta) \neq 0$ for $m < k$, so that $\sin\left(\frac{1}{2}k\theta\right) \neq 0$

and hence $\cos\left(\frac{1}{2}k\theta\right) = 0$

3rd Part

Suppose that $\theta = \frac{c}{d}$ is rational (where c & d are positive integers). Then it would be possible to find a positive integer k such that $\sin(k\theta) = 0$, such that $\sin(m\theta) \neq 0$ for $0 < m < k$ (Let k be the smallest positive integer such that $k\frac{c}{d} = 180n$, for some positive integer n. Such a k will exist, as $(180d).\frac{c}{d} = 180c$, and we could then consider each positive integer less than $180d$.)

From the 2nd Part of (i),

$cos(k\theta) = cos^k\theta(1 - \binom{k}{2}(sec^2\theta - 1) + \binom{k}{4}(sec^2\theta - 1)^2 - \cdots)$

and so (as $\frac{k}{2}$ is an integer)

$cos\left(\frac{k}{2}\theta\right) = cos^{(\frac{k}{2})}\theta(1 - \binom{\frac{k}{2}}{2}(a^2 - 1) + \binom{\frac{k}{2}}{4}(a^2 - 1)^2 - \cdots)$

Then $cos^{(\frac{k}{2})}\theta = (\frac{1}{a})^{(\frac{k}{2})} \neq 0$

And $a^2 - 1$ is even (as a is odd), so that $\binom{\frac{k}{2}}{2}(a^2 - 1)$ is an even integer. Also $(a^2 - 1)^2$ is even, so that $\binom{\frac{k}{2}}{4}(a^2 - 1)^2$ is also an even integer, and so on for the other terms.

Thus, $1 - \binom{\frac{k}{2}}{2}(a^2 - 1) + \binom{\frac{k}{2}}{4}(a^2 - 1)^2 - \cdots$ is odd, and hence can't be zero.

Therefore $cos\left(\frac{k}{2}\theta\right) \neq 0$, contradicting the result in the 2nd Part.

And so θ must be irrational, as required.

(iii) As in (ii), write

$sin(k\phi) = sin\phi cos^{k-1}\phi \left(k - \binom{k}{3}(sec^2\phi - 1)\right.$

$\left. + \binom{k}{5}(sec^2\phi - 1)^2 - \cdots \right)$ (***)

Now, $\phi = cot^{-1}\left(\frac{1}{b}\right) \Rightarrow cot\phi = \frac{cos\phi}{sin\phi} = \frac{1}{b}$, so that $sin\phi \neq 0$, otherwise $cos\phi = 1$ and $cot\phi$ is undefined.

Also $cos\phi \neq 0$, otherwise $sin\phi = 1$ and $cot\phi = 0$.

And $sec^2\phi - 1 = tan^2\phi = b^2$, which is even.

As before, if ϕ is rational, then $sin(k\phi) = 0$ for some positive integer k, such that $sin(m\phi) \neq 0$ for all integer m such that $0 < m < k$.

Then, from (***), $sin(k\phi) = 0$

$\Rightarrow k - \binom{k}{3}(sec^2\phi - 1) + \binom{k}{5}(sec^2\phi - 1)^2 - \cdots = 0$,

as $sin\phi \neq 0$ & $cos\phi \neq 0$,

ie $k - \binom{k}{3}b^2 + \binom{k}{5}b^4 - \cdots = 0$

and so, as all the terms after the 1st are even, it follows that k must be even.

As before, $sin(k\phi) = 0 \Rightarrow 2\sin\left(\frac{1}{2}k\phi\right)\cos\left(\frac{1}{2}k\phi\right) = 0$

As $\sin(m\phi) \neq 0$ for $m < k$, $\sin\left(\frac{1}{2}k\phi\right) \neq 0$, and hence $\cos\left(\frac{1}{2}k\phi\right) = 0$

Then, from the 2nd Part of (i),

$\cos\left(\frac{1}{2}k\phi\right)$

$= cos^{\left(\frac{1}{2}k\right)}\phi(1 - \binom{\frac{1}{2}k}{2}(sec^2\phi - 1) + \binom{\frac{1}{2}k}{4}(sec^2\phi - 1)^2 - \cdots)$

$= 0$

Hence, as $cos\phi \neq 0$ and $sec^2\phi - 1 = b^2$,

$1 - \binom{\frac{1}{2}k}{2}b^2 + \binom{\frac{1}{2}k}{4}b^4 - \cdots = 0$

But, as b^2, b^4 etc are even, this is impossible (as the LHS is $odd - even + even - \cdots = odd$).

So this contradicts the supposition that ϕ is rational, and so ϕ must be irrational, as required.