STEP 2022, P3, Q5 - Solution (6 pages; 6/2/24)

(i) $I = \int_{-a}^{a} \frac{1}{1+e^x}\, dx = \int_{-a}^{a} \frac{e^{-x}}{e^{-x}+1}\, dx$ [in order to obtain an

integrand of the form $f'(x)g\big(f(x)\big)$, so that the substitution

$u = f(x)$ works]

Let $u = e^{-x}$, so that $du = -e^{-x}dx$,

and $I = \int_{e^a}^{e^{-a}} \frac{-1}{u+1}\, du = -[\ln(u+1)]_{e^a}^{e^{-a}}$

[noting that $u + 1 = e^{-x} + 1 > 0$, so that $\ln(u+1)$ is defined]

$= \ln(e^a + 1) - \ln(e^{-a} + 1)$

$= \ln\left(\frac{e^a+1}{e^{-a}+1}\right)$

$= \ln\left(\frac{e^a(e^a+1)}{1+e^a}\right)$

$= \ln(e^a) = a$

[This result is in fact also true when $a < 0$.]

(ii) **1st Part**

Suppose that $g(x) \neq 0$ for some $x \geq 0$. (*)

Then, as $g(x)$ is a continuous function, there will be an $\varepsilon > 0$ such

that $g(x)$ has the same sign for all x such that $x_1 \leq x \leq x_1 + \varepsilon$,

so that $\int_{x_1}^{x_1+\varepsilon} g(x)dx \neq 0$ (**)

(considering the integral as the area under the graph of $g(x)$)

But $\int_{x_1}^{x_1+\varepsilon} g(x)dx = \int_0^{x_1+\varepsilon} g(x)dx - \int_0^{x_1} g(x)dx = 0 - 0$,

as $\int_0^a g(x)dx = 0$ for all $a \geq 0$,

and this contradicts (**).

Hence (*) cannot be true, and so $g(x) = 0$ for all $x \geq 0$.

2nd Part

Let $g(x) = \frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1$ [in order to use the 1st part, hopefully]

As $f(x) \geq 0$, $\frac{1}{1+f(x)}$ and $\frac{1}{1+f(-x)}$ are defined for all x. And therefore, as $f(x)$ is continuous, $g(x)$ will also be continuous.

Then $\int_{-a}^{a} \frac{1}{1+f(x)}\, dx = a$ (for all $a \geq 0$)

$\Rightarrow \int_{-a}^{a} g(x)\, dx = \int_{-a}^{a} \frac{1}{1+f(x)}\, dx + \int_{-a}^{a} \frac{1}{1+f(-x)}\, dx + \int_{-a}^{a} -1\, dx$

Writing $u = -x$, the 2nd integral $= \int_a^{-a} \frac{1}{1+f(u)}(-1)du$

$= \int_{-a}^{a} \frac{1}{1+f(x)}\, dx$,

so that $\int_{-a}^{a} g(x)dx = 2a + [-x]_{-a}^{a} = 2a - a - a = 0$ (***)

Then, as $g(x) = \frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1$, $g(x)$ is an even function

[ie $g(-x) = g(x)$], and so $\int_{-a}^{0} g(x)dx = \int_{0}^{a} g(x)dx$,

and then from (***):

$0 = \int_{-a}^{a} g(x)dx = \int_{-a}^{0} g(x)dx + \int_{0}^{a} g(x)dx = 2\int_{0}^{a} g(x)dx$,

so that $\int_{0}^{a} g(x)dx = 0$ (for all $a \geq 0$).

So $\int_{-a}^{a} \frac{1}{1+f(x)}\, dx = a$ (for all $a \geq 0$) $\Rightarrow \int_{-a}^{a} g(x)dx = 0$

$\Rightarrow \int_{0}^{a} g(x)dx = 0$ (for all $a \geq 0$)

$\Rightarrow g(x) = 0$ for all $x \geq 0$ (from the result in the 1st Part)

ie $\frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1 = 0$ for all $x \geq 0$

This is the 'only if' part of the required result.

For the 'If' part:

Suppose that $\frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1 = 0$ (for all $x \geq 0$)

Then, as $\int_{-a}^{a} \frac{1}{1+f(-x)}\, dx = \int_{-a}^{a} \frac{1}{1+f(x)}\, dx$,

$\int_{-a}^{a} \frac{1}{1+f(x)}\, dx = \frac{1}{2}\{\int_{-a}^{a} \frac{1}{1+f(x)}\, dx + \int_{-a}^{a} \frac{1}{1+f(-x)}\, dx\}$

$= \frac{1}{2}\int_{-a}^{a} 1\, dx = \frac{1}{2}\big(a - (-a)\big) = a$, as required (with no restriction on a, and so it applies when $a \geq 0$)

3rd Part

Result to prove:

$\frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1 = 0$ (for all $x \geq 0$)

if and only if $f(x)f(-x) = 1$ for all x

Only if part:

$\frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1 = 0$ (for all $x \geq 0$) $\Rightarrow \frac{[1+f(-x)]+[1+f(x)]}{[1+f(x)][1+f(-x)]} = 1$

$\Rightarrow 2 + f(-x) + f(x) = 1 + f(-x) + f(x) + f(x)f(-x)$

$\Rightarrow 1 = f(x)f(-x)$ (for all $x \geq 0$)

Also, writing $y = -x$, where $x > 0$ (so that $y < 0$):

$1 = f(-y)f(y) = f(y)f(-y)$,

so that $1 = f(x)f(-x)$ is also true when $x < 0$

If part:

$f(x)f(-x) = 1$ (for all x)

$\Rightarrow \frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1 = \frac{1}{1+f(x)} + \frac{1}{1+\frac{1}{f(x)}} - 1$

$= \frac{1}{1+f(x)} + \frac{f(x)}{f(x)+1} - 1$

$= \frac{1+f(x)}{1+f(x)} - 1 = 0$ (for all x)

and so $\frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1 = 0$ for all $x \geq 0$, in particular

(iii) From the 3rd Part of (ii), $\int_{-a}^{a} \frac{1}{1+f(x)}\, dx = a$ for all $a \geq 0$

Then $\int_{-a}^{a} \frac{h(x)}{1+f(x)}\, dx = \int_{-a}^{0} \frac{h(x)}{1+f(x)}\, dx + \int_{0}^{a} \frac{h(x)}{1+f(x)}\, dx$

For the 1st integral, let $u = -x$, so that

$\int_{-a}^{0} \frac{h(x)}{1+f(x)}\, dx = \int_{a}^{0} \frac{h(-u)}{1+f(-u)}\, (-1)du$

$= \int_{0}^{a} \frac{h(u)}{1+\frac{1}{f(u)}} du = \int_{0}^{a} \frac{f(u)h(u)}{f(u)+1} du,$

and so $\int_{-a}^{a} \frac{h(x)}{1+f(x)}\, dx = \int_{0}^{a} \frac{f(x)h(x)+h(x)}{1+f(x)}\, dx$

$= \int_{0}^{a} h(x)\, dx$, as required.

(iv) [We want to write $\frac{e^{-x}cosx}{coshx}$ in the form $\frac{h(x)}{1+f(x)}$, where

$h(-x) = h(x)$ and $f(-x) = \frac{1}{f(x)}$, and $f(x) \geq 0$;

so setting $e^{-x}cosx = h(x)$ (the most obvious thing to try) doesn't work, as $h(-x) \neq h(x)$;

so we can try to rearrange $\frac{e^{-x}cosx}{coshx}$]

$\frac{e^{-x}cosx}{coshx} = \frac{2cosx}{e^{x}(e^{x}+e^{-x})} = \frac{2cosx}{e^{2x}+1}$,

and we can set $h(x) = 2cosx$ and $f(x) = e^{2x}$, as then $h(-x) = h(x)$ and $f(-x) = \frac{1}{f(x)}$, with $f(x) \geq 0$

So, from (iii): $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2cosx}{e^{2x}+1}\, dx = \int_{0}^{\frac{\pi}{2}} 2cosx\, dx = 2[sinx]_{0}^{\frac{\pi}{2}} = 2$