STEP 2022, P3, Q11 - Solution (5 pages; 15/2/24)

(i)(a) $P(X \leq n-1) + P(X = n) + P(X \geq n+1) = 1$ (*)

$\mu = N\left(\frac{1}{2}\right) = n$, and so by symmetry (being a Binomial distribution with probability $\frac{1}{2}$): $P(X \geq n+1) = P(X \leq n-1)$

Hence (*) $\Rightarrow 2P(X \leq n-1) + P(X = n) = 1$

$\Rightarrow P(X \leq n-1) = \frac{1}{2}\big(1 - P(X = n)\big)$, as required.

(b) $\delta = \sum_{r=0}^{2n} |r-n| \binom{2n}{r} (\frac{1}{2})^{2n}$

By symmetry,

$\sum_{r=0}^{n-1} |r-n| \binom{2n}{r} (\frac{1}{2})^{2n} = \sum_{r=n+1}^{2n} |r-n| \binom{2n}{r} (\frac{1}{2})^{2n}$,

and $|n-n| \binom{2n}{r} \left(\frac{1}{2}\right)^{2n} = 0$,

so that $\delta = 2\sum_{r=0}^{n-1} |r-n| \binom{2n}{r} (\frac{1}{2})^{2n}$

$= \sum_{r=0}^{n-1} (n-r) \binom{2n}{r} \frac{1}{2^{2n-1}}$, as required.

(c) **1st Part**

$r\binom{2n}{r} = r\frac{(2n)!}{r!(2n-r)!} = 2n\frac{(2n-1)!}{(r-1)!([2n-1]-[r-1])} = 2n\binom{2n-1}{r-1}$,

as required

2nd Part

$\delta = \sum_{r=0}^{n-1}(n-r)\binom{2n}{r}\frac{1}{2^{2n-1}}$

$= \frac{1}{2^{2n-1}}\sum_{r=0}^{n-1} n\binom{2n}{r} - \frac{1}{2^{2n-1}}\sum_{r=1}^{n-1} r\binom{2n}{r}$

$= \frac{1}{2^{2n-1}}\sum_{r=0}^{n-1} n\binom{2n}{r} - \frac{1}{2^{2n-1}}\sum_{r=1}^{n-1} 2n\binom{2n-1}{r-1}$, (**)

from the 1st Part of (c).

Now $\sum_{r=0}^{2n}\binom{2n}{r} = (1+1)^{2n} = 2^{2n}$

Also, $\sum_{r=0}^{2n}\binom{2n}{r}$

$= [\sum_{r=0}^{n-1}\binom{2n}{r}] + \binom{2n}{n} + [\sum_{r=n+1}^{2n}\binom{2n}{r}]$

$= 2[\sum_{r=0}^{n-1}\binom{2n}{r}] + \binom{2n}{n}$, by the symmetry of Pascal's triangle

Hence, $\sum_{r=0}^{n-1}\binom{2n}{r} = \frac{1}{2}[2^{2n} - \binom{2n}{n}]$

So, from (**), $\delta = \frac{n}{2^{2n-1}}[\sum_{r=0}^{n-1}\binom{2n}{r} - 2\sum_{r=1}^{n-1}\binom{2n-1}{r-1}]$

$= \frac{n}{2^{2n-1}}[\sum_{r=0}^{n-1}\binom{2n}{r} - 2\sum_{r-1=0}^{n-2}\binom{2n-1}{r-1}]$

$= \frac{n}{2^{2n-1}}[\sum_{r=0}^{n-1}\binom{2n}{r} - 2\sum_{R=0}^{n-2}\binom{2n-1}{R}]$

or $\frac{n}{2^{2n-1}}[\sum_{r=0}^{n-1}\binom{2n}{r} - 2[[\sum_{r=0}^{n-1}\binom{2n-1}{r}] - \binom{2n-1}{n-1}]$

Then, by the symmetry of Pascal's triangle again,

$\sum_{r=0}^{2n-1}\binom{2n-1}{r} = [\sum_{r=0}^{n-1}\binom{2n-1}{r}] + [\sum_{r=n}^{2n-1}\binom{2n-1}{r}]$

$= 2\sum_{r=0}^{n-1}\binom{2n-1}{r}$,

so that $(1+1)^{2n-1} = 2\sum_{r=0}^{n-1}\binom{2n-1}{r}$,

and hence $\delta = \frac{n}{2^{2n-1}}[\frac{1}{2}[2^{2n} - \binom{2n}{n}] - 2^{2n-1} + 2\binom{2n-1}{n-1}]$

$= \frac{n}{2^{2n-1}}[2\binom{2n-1}{n-1} - \frac{1}{2}\binom{2n}{n}]$

$= \frac{n}{2^{2n-1}}[\frac{2(2n-1)!}{(n-1)!([2n-1]-[n-1])!} - \frac{(2n)!}{2n!n!}]$

$= \frac{n}{2^{2n-1}}[\frac{4(2n-1)!}{2(n-1)!n!} - \frac{(2n)!}{2n!n!}]$

$= \frac{n}{2^{2n-1}} \cdot \frac{(2n-1)!(4n-2n)}{2n!n!}$

$= \frac{n}{2^{2n-1}} \cdot \frac{(2n)!}{2n!n!}$

$= \frac{n}{2^{2n}}\binom{2n}{n}$, as required.

(ii) Now $\mu = N\left(\frac{1}{2}\right) = (2n+1)\left(\frac{1}{2}\right) = n + \frac{1}{2}$

$\delta = \sum_{r=0}^{2n+1}|r - n - \frac{1}{2}|\binom{2n+1}{r}(\frac{1}{2})^{2n+1}$

By symmetry,

$\sum_{r=0}^{n}|r - n - \frac{1}{2}|\binom{2n+1}{r}(\frac{1}{2})^{2n+1}$

$= \sum_{r=n+1}^{2n+1}|r - n - \frac{1}{2}|\binom{2n+1}{r}(\frac{1}{2})^{2n+1}$,

so that $\delta = \frac{2}{2^{2n+1}}\sum_{r=0}^{n}(n + \frac{1}{2} - r)\binom{2n+1}{r}$

Then $r\binom{2n+1}{r} = r\frac{(2n+1)!}{r!(2n+1-r)!}$

$= (2n+1)\frac{(2n)!}{(r-1)!(2n-[r-1])} = (2n+1)\binom{2n}{r-1}$,

and so $\delta = \frac{(n+\frac{1}{2})}{2^{2n}}[\sum_{r=0}^{n}\binom{2n+1}{r}] - \frac{(2n+1)}{2^{2n}}[\sum_{r=1}^{n}\binom{2n}{r-1}]$

(as the term $r\binom{2n+1}{r}$ is zero when $r = 0$)

$= \frac{(n+\frac{1}{2})}{2^{2n}}[\sum_{r=0}^{n}\binom{2n+1}{r}] - \frac{(2n+1)}{2^{2n}}[\sum_{r=0}^{n-1}\binom{2n}{r}]$ (***)

Also, $\sum_{r=0}^{2n}\binom{2n+1}{r}$ (which equals $(1+1)^{2n+1}$)

$= [\sum_{r=0}^{n}\binom{2n+1}{r}] + [\sum_{r=n+1}^{2n+1}\binom{2n+1}{r}]$

$= 2[\sum_{r=0}^{n}\binom{2n+1}{r}]$, by the symmetry of Pascal's triangle

Hence, $\sum_{r=0}^{n}\binom{2n+1}{r} = \frac{1}{2}[2^{2n+1}] = 2^{2n}$

And $\sum_{r=0}^{2n}\binom{2n}{r}$ (which equals $(1+1)^{2n}$)

$= [\sum_{r=0}^{n-1}\binom{2n}{r}] + \binom{2n}{n} + [\sum_{r=n+1}^{2n}\binom{2n}{r}]$

$= 2[\sum_{r=0}^{n-1}\binom{2n}{r}] + \binom{2n}{n}$, by the symmetry of Pascal's triangle

Hence, $\sum_{r=0}^{n-1}\binom{2n}{r} = \frac{1}{2}[2^{2n} - \binom{2n}{n}]$

Then, from (***), $\delta = \frac{(n+\frac{1}{2})}{2^{2n}}[\sum_{r=0}^{n}\binom{2n+1}{r}] - \frac{(2n+1)}{2^{2n}}[\sum_{r=0}^{n-1}\binom{2n}{r}]$

$= \frac{(n+\frac{1}{2})}{2^{2n}}[2^{2n}] - \frac{(2n+1)}{2^{2n}}[\frac{1}{2}[2^{2n} - \binom{2n}{n}]]$

$= \left(n + \frac{1}{2} - \frac{1}{2}(2n+1)\right) + \frac{(2n+1)}{2^{2n+1}}\binom{2n}{n}$

$= \frac{(2n+1)}{2^{2n+1}}\binom{2n}{n}$