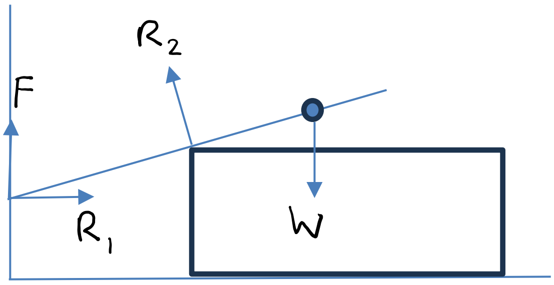
STEP 2022, P2, Q9 - Solution (4 pages; 21/7/23)

(i)



Referring to the diagram:

Resolving horiz. : $R_1 + R_2\cos(90° + \theta) = 0$,

so that $R_1 = -R_2(-sin\theta) = R_2 sin\theta$

Taking moments about P,

rotational equilibrium $\Rightarrow R_2(x - dsec\theta) + Fxcos\theta = R_1 xsin\theta$

$\Rightarrow Fxcos\theta = R_2(xsin^2\theta - x + dsec\theta)$

$= R_2(dsec\theta - xcos^2\theta)$

Then if $x = dsec^3\theta$,

$Fxcos\theta = R_2(dsec\theta - dsec\theta) = 0$, so that $F = 0$, as required

(ii) **1st Part**

$F \leq \mu R_1$

And from (i), $Fxcos\theta = R_2(dsec\theta - xcos^2\theta)$

$= \frac{R_1}{sin\theta}(dsec\theta - xcos^2\theta)$, on the provisional assumption that F is in the upwards direction.

Hence $\mu \geq \frac{F}{R_1} = \frac{dsec\theta - xcos^2\theta}{xsin\theta cos\theta} = \frac{dsec^3\theta - x}{xtan\theta}$

However, as it is given that $x > dsec^3\theta$, we now see that this implies a negative F, and so the friction must in fact be in the downwards direction, giving

$-Fxcos\theta = \frac{R_1}{sin\theta}(dsec\theta - xcos^2\theta)$,

so that $\mu \geq \frac{F}{R_1} = \frac{xcos^2\theta - dsec\theta}{xsin\theta cos\theta} = \frac{x - dsec^3\theta}{xtan\theta}$,

and thus $\mu \geq \frac{x - dsec^3\theta}{xtan\theta}$, as required.

2nd Part

If instead $x < dsec^3\theta$, then $\mu \geq \frac{dsec^3\theta - x}{xtan\theta}$, as above.

(iii) <u>Case 1: $x > dsec^3\theta$</u>

$\mu \geq \frac{x - dsec^3\theta}{xtan\theta}$,

so that $\mu tan\theta \geq 1 - \left(\frac{d}{x}\right)sec^3\theta$,

and $\left(\frac{d}{x}\right)sec^3\theta \geq 1 - \mu tan\theta$,

If $\mu < cot\theta$, then $\mu tan\theta < 1$ and $1 - \mu tan\theta > 0$,

so that $\frac{sec^3\theta}{1 - \mu tan\theta} \geq \frac{x}{d}$ (*)

Also, $\frac{x}{d} > sec^3\theta > \frac{sec^3\theta}{1+\mu tan\theta}$,

so that $\frac{x}{d} \geq \frac{sec^3\theta}{1+\mu tan\theta}$ (**)

Case 2: $x < dsec^3\theta$

$\mu \geq \frac{dsec^3\theta - x}{xtan\theta}$,

then $\mu tan\theta \geq \left(\frac{d}{x}\right) sec^3\theta - 1$,

and $1 + \mu tan\theta \geq \left(\frac{d}{x}\right) sec^3\theta$,

so that $\frac{x}{d} \geq \frac{sec^3\theta}{1+\mu tan\theta}$ (***)

Also, $\frac{x}{d} < sec^3\theta < \frac{sec^3\theta}{1-\mu tan\theta}$, provided that $1 - \mu tan\theta > 0$;

so that $\frac{x}{d} \leq \frac{sec^3\theta}{1-\mu tan\theta}$, provided that $\mu < cot\theta$ (****)

Case 3: $x = dsec^3\theta$

$\frac{x}{d} = sec^3\theta \geq \frac{sec^3\theta}{1+\mu tan\theta}$ (*****)

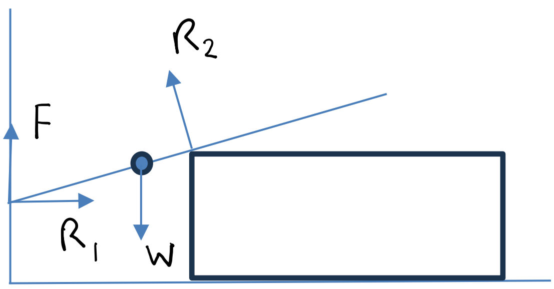
And $\frac{x}{d} = sec^3\theta \leq \frac{sec^3\theta}{1-\mu tan\theta}$, provided that $1 - \mu tan\theta > 0$, and so $\mu < cot\theta$ (******)

Conclusion

In all 3 cases, $\frac{x}{d} \geq \frac{sec^3\theta}{1+\mu tan\theta}$ (from (**) , (***) & (*****))

Also, in all 3 cases, $\frac{x}{d} \leq \frac{sec^3\theta}{1-\mu tan\theta}$, with the further condition in all 3 cases that $\mu < cot\theta$ (from (*), (****) & (******))

(iv)



In this case, $x < dsec\theta < dsec^3\theta$

Referring to the diagram above,

rotational equilibrium $\Rightarrow R_2(dsec\theta - x) + R_1 xsin\theta = Fxcos\theta$

(in this situation, the frictional force has to be upwards; otherwise there will be a positive anti-clockwise moment about P)

$\Rightarrow Fxcos\theta = R_2(dsec\theta - x + xsin^2\theta) = R_2(dsec\theta - xcos^2\theta)$

and $\mu \geq \frac{dsec^3\theta - x}{xtan\theta}$ as in the 2nd Part of (ii)

Then $\mu xtan\theta \geq dsec^3\theta - x$,

and hence $x(\mu tan\theta + 1) \geq dsec^3\theta$,

so that $x \geq \frac{dsec^3\theta}{\mu tan\theta + 1}$

Then, as $x < dsec\theta$, $\frac{dsec^3\theta}{\mu tan\theta + 1} \leq x < dsec\theta$,

so that $sec^2\theta < \mu tan\theta + 1$,

and hence $\mu tan\theta > sec^2\theta - 1 = tan^2\theta$,

so that $\mu > tan\theta$, as required.