STEP 2022, P2, Q8 - Solution (3 pages; 16/7/23)

(i) **1st Part**

Let the invariant lines be $y = mx$ (for two values of m) (which excludes the y axis).

Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} ax + bmx \\ cx + dmx \end{pmatrix}$

and $cx + dmx = m(ax + bmx)$,

so that $c + dm = ma + bm^2$

or $bm^2 + (a - d)m - c = 0$, as required

2nd Part

If one invariant line is the y axis,

then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} by \\ dy \end{pmatrix}$

and $by = 0$ for all y, so that $b = 0$

Suppose that the other invariant line is $y = mx$.

As before, $bm^2 + (a - d)m - c = 0$,

and so, as $b = 0$, $m = \frac{c}{a-d}$ (noting that $a \neq d$).

(ii) Case 1: One of the invariant lines is the y axis, so that the other line is $y = \frac{c}{a-d}\, x$

Result to prove: $(a - d)^2 = (b - c)^2 - 4bc = c^2$

If the angle between the lines is 45°, then $\frac{c}{a-d} = 1$, so that

$c = a - d$, and hence $(a-d)^2 = c^2$, as required.

Case 2: Neither invariant line is the y axis, so that

$bm^2 + (a-d)m - c = 0$, with roots m_1 & m_2, say. (*)

The direction vectors for the lines are $\begin{pmatrix} 1 \\ m_1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ m_2 \end{pmatrix}$,

and if the angle between the lines is 45°, then

$\begin{pmatrix} 1 \\ m_1 \end{pmatrix} . \begin{pmatrix} 1 \\ m_2 \end{pmatrix} = \left| \begin{pmatrix} 1 \\ m_1 \end{pmatrix} \right| . \left| \begin{pmatrix} 1 \\ m_2 \end{pmatrix} \right| cos 45°$,

so that $1 + m_1 m_2 = \sqrt{1 + {m_1}^2} . \sqrt{1 + {m_2}^2} . \frac{1}{\sqrt{2}}$

and hence $2(1 + m_1 m_2)^2 = (1 + {m_1}^2)(1 + {m_2}^2)$,

so that $(m_1 m_2)^2 + 4m_1 m_2 + 1 - {m_1}^2 - {m_2}^2 = 0$ (**)

From (*), $m_1 m_2 = -\frac{c}{b}$

Also $b{m_1}^2 + (a-d)m_1 - c = 0$ and $b{m_2}^2 + (a-d)m_2 - c = 0$,

so that $b({m_1}^2 + {m_2}^2) = 2c - (a-d).(-\frac{a-d}{b})$

Then (**) becomes $(-\frac{c}{b})^2 + 4\left(-\frac{c}{b}\right) + 1 - \frac{1}{b}\left[2c + \frac{(a-d)^2}{b}\right] = 0$

or $c^2 - 4bc + b^2 - 2bc - (a-d)^2 = 0$,

so that $(a-d)^2 = (b-c)^2 - 4bc$, as required.

(iii) [Note that the matrix is such that there are two distinct invariant lines passing through the Origin.]

Case 1: One of the invariant lines is the y axis

Then the other invariant line has to be the x axis; ie $m = 0$.

From (i), $m = \frac{c}{a-d}$, so that $c = 0$

Also, from (i), $b = 0$

Case 2: Neither invariant line is the y axis

The two lines will make equal angles with $y = x$ if $m_2 = \frac{1}{m_1}$; ie if

$m_1 m_2 = 1$

[The lines are distinct, so we don't have to consider the case $m_1 = m_2$.]

From (i), $bm^2 + (a-d)m - c = 0$,

If $m_1 m_2 = 1$, then $\frac{-c}{b} = 1$, so that $c = -b$ or $b + c = 0$

Thus, necessary and sufficient conditions are:

$b = \ c = 0$ or $b + c = 0$;

in other words, $b + c = 0$

(iv) $(a-d)^2 = (b-c)^2 - 4bc$ and $b + c = 0$

$\Rightarrow (a-d)^2 = 4b^2 + 4b^2 = 8b^2$

eg $a = 0, b = 1, c = -1, d = 2\sqrt{2}$; ie $\begin{pmatrix} 0 & 1 \\ -1 & 2\sqrt{2} \end{pmatrix}$