STEP 2022, P2, Q2 - Solution (4 pages; 14/3/24)

(i) **1st Part**

$u_{n+1} = \frac{1}{2}(u_{n+2} + u_n) \Rightarrow 2u_{n+1} = u_{n+2} + u_n$

$\Rightarrow u_{n+2} - u_{n+1} = u_{n+1} - u_n$ for all $n \geq 1$;

ie there is a constant difference between successive terms, as required.

Let $u_n = f(n)$, where $f(n)$ is a polynomial in n.

As $u_{n+1} - u_n = c$, a constant,

2nd Part

Consider $a(n+1)^3 - an^3 = 3an^2 + \cdots$

Then, if $u_n = f(n) = an^r + bn^{r-1} + \cdots$, where $a \neq 0$ and $r \geq 1$,

$u_{n+1} - u_n = [a(n+1)^r - an^r] + [a(n+1)^{r-1} - an^{r-1}] + \cdots$

$= ran^{r-1} + \cdots$

and in order that $u_{n+1} - u_n = c$, it follows that $r - 1 = 0$,

so that the degree of u_n is 1, unless $r = 0$, when the degree is 0;

so the degree is at most 1, as required.

(ii) **1st Part**

$v_{n+1} = \frac{1}{2}(v_{n+2} + v_n) - p$

Define t_n by $v_n = t_n + pn^2$, so that

$t_{n+1} + p(n+1)^2 = \frac{1}{2}([t_{n+2} + p(n+2)^2] + [t_n + pn^2]) - p,$

and hence

$t_{n+1} = \frac{1}{2}(t_{n+2} + t_n) + \frac{p}{2}[(n+2)^2 + n^2 - 2 - 2(n+1)^2]$

$= \frac{1}{2}(t_{n+2} + t_n) + \frac{p}{2}(0)$

Thus, from (i), t_n has degree at most 1,

so that t_n can be written in the form $an + b$,

and then $v_n = t_n + pn^2 = pn^2 + an + b$; ie v_n has degree 2 (as $p \neq 0$), as required.

2nd Part

$v_1 = 0 \Rightarrow p + a + b = 0$ (1)

$v_2 = 0 \Rightarrow 4p + 2a + b = 0$ (2)

Then $(2) - (1) \Rightarrow 3p + a = 0; a = -3p,$

and then from (1), $b = -p - (-3p) = 2p$

Thus $v_n = pn^2 - 3pn + 2p = p(n-1)(n-2)$

(iii) **1st Part**

$w_{n+1} = \frac{1}{2}(w_{n+2} + w_n) - an - b$ (*)

[Based on the method for (ii):]

Define T_n by $w_n = T_n + An^3 + Bn^2$.

Then (*) $\Rightarrow$ $w_{n+1} = T_{n+1} + A(n+1)^3 + B(n+1)^2$

$= \frac{1}{2}([T_{n+2} + A(n+2)^3 + B(n+2)^2] + [T_n + An^3 + Bn^2])$

$-an - b$, so that

$$T_{n+1} = \frac{1}{2}(T_{n+2} + T_n) + \frac{n^3}{2}(2A - 2A) + \frac{n^2}{2}(6A + 2B - 6A - 2B)$$

$$+\frac{n}{2}(12A + 4B - 6A - 4B - 2a) + \frac{1}{2}(8A + 4B - 2A - 2B - 2b)$$

Then, setting $12A + 4B - 6A - 4B - 2a = 0$

and $8A + 4B - 2A - 2B - 2b = 0$ gives:

$6A - 2a = 0$ & $6A + 2B - 2b = 0$,

so that we need $A = \frac{a}{3}$ & $2B = 2b - 6A = 2b - 2a$; ie $B = b - a$

Then $T_{n+1} = \frac{1}{2}(T_{n+2} + T_n)$, and by (i) again T_n has degree at most 1, so that w_n can be written in the form $w_n = T_n + An^3 + Bn^2$

$= C + Dn + \frac{a}{3}n^3 + (b - a)n^2$; ie w_n has degree 3

2nd Part

Given that $w_1 = w_2 = 0$,

$C + D + \frac{a}{3} + (b - a) = 0$ (1) & $C + 2D + \frac{8a}{3} + 4(b - a) = 0$ (2)

Then (2) − (1) gives $D + \frac{7a}{3} + 3(b - a) = 0$,

so that $D = \frac{-7a - 9b + 9a}{3} = \frac{2a - 9b}{3}$,

and then (1) gives $C + \frac{2a - 9b}{3} + \frac{a}{3} + (b - a) = 0$,

so that $C = \frac{1}{3}\left(-2a + 9b - a - 3(b - a)\right) = \frac{1}{3}(6b) = 2b$

and then $w_n = C + Dn + \frac{a}{3}n^3 + (b - a)n^2$

$= 2b + \frac{2a - 9b}{3}n + (b - a)n^2 + \frac{a}{3}n^3$

[Note: Here we have extended the method indicated in part (ii) (ie writing $w_n = T_n + An^3 + Bn^2$). The official sol'n employs a simpler extension of the method, writing $w_n = T_n + An^3$, but the price that is paid for this simpler approach is having to consider two separate cases, depending on whether $b = a$ (if it does, then the result from (i) is used; otherwise the result from (ii) is used).]