STEP 2022, P2, Q12 - Solution (4 pages; 19/2/24)

(i) **1st Part**

$\int_0^1 kx^n(1-x)dx = 1,$

so that $k\left[\frac{1}{n+1}x^{n+1} - \frac{1}{n+2}x^{n+2}\right]_0^1 = 1$

$\Rightarrow k\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = 1$

$\Rightarrow \frac{k}{(n+1)(n+2)} = 1$

and $k = (n+1)(n+2)$, as required.

2nd Part

$\mu = \int_0^1 x.kx^n(1-x)dx$

$= k\left[\frac{1}{n+2}x^{n+2} - \frac{1}{n+3}x^{n+3}\right]_0^1$

$= k\left(\frac{1}{n+2} - \frac{1}{n+3}\right)$

$= \frac{k}{(n+2)(n+3)}$

$= \frac{(n+1)(n+2)}{(n+2)(n+3)}$

$= \frac{(n+1)}{(n+3)}$

[Note that $0 < \mu < 1$, and that $\mu \to 1$ as $n \to \infty$, as expected.]

(ii) **1st Part**

Let m be the median of X.

Then $\int_0^m kx^n(1-x)dx = \frac{1}{2}$,

so that $k\left[\frac{1}{n+1}x^{n+1} - \frac{1}{n+2}x^{n+2}\right]\begin{matrix} m \\ 0 \end{matrix} = \frac{1}{2}$

$\Rightarrow k\left(\frac{1}{n+1}m^{n+1} - \frac{1}{n+2}m^{n+2}\right) = \frac{1}{2}$

and $(n+2)m^{n+1} - (n+1)m^{n+2} = \frac{1}{2}$

Now, the LHS of this equation is an increasing function of m

(being the probability that $X \le m$), and so when $m > \mu$,

$(n+2)m^{n+1} - (n+1)m^{n+2} > (n+2)\mu^{n+1} - (n+1)\mu^{n+2}$

and v.v.

ie $m > \mu \Leftrightarrow \frac{1}{2} > (n+2)[\frac{(n+1)}{(n+3)}]^{n+1} - (n+1)[\frac{(n+1)}{(n+3)}]^{n+2}$

$\Leftrightarrow (n+3)^{n+2}$

$> 2(n+2)(n+3)(n+1)^{n+1} - 2(n+1)^{n+3}$

$\Leftrightarrow (n+3)^{n+2} > 2(n+1)^{n+1}[(n+2)(n+3) - (n+1)^2]$

$\Leftrightarrow (n+3)^{n+2} > 2(n+1)^{n+1}[3n+5]$

$\Leftrightarrow (n+3)\left(\frac{n+3}{n+1}\right)^{n+1} > 2[3n+5]$

$\Leftrightarrow \left(1+\frac{2}{n+1}\right)^{n+1} > \frac{2(3n+5)}{n+3} = \frac{6(n+3)-8}{n+3} = 6 - \frac{8}{n+3}$

Thus, $\left(1+\frac{2}{n+1}\right)^{n+1} > 6 - \frac{8}{n+3} \Rightarrow m > \mu$, as required.

2nd Part

$\left(1+\frac{2}{n+1}\right)^{n+1}$

$> 1+(n+1)\left(\frac{2}{n+1}\right)+\binom{n+1}{2}(\frac{2}{n+1})^2+\binom{n+1}{3}(\frac{2}{n+1})^3$

$= 1+2+\frac{2n}{n+1}+\frac{4n(n-1)}{3(n+1)^2}$

$= \frac{9(n+1)^2+6n(n+1)+4n(n-1)}{3(n+1)^2}$

$= \frac{19n^2+20n+9}{3(n+1)^2}$

$= \frac{18(n+1)^2+n^2-16n-9}{3(n+1)^2}$

$= 6+\frac{n^2-16n-9}{3(n+1)^2}$

Thus $m > \mu$ provided that $\frac{n^2-16n-9}{3(n+1)^2} > -\frac{8}{n+3}$

ie provided that $(n^2-16n-9)(n+3)+24(n+1)^2 > 0$ (*)

Then LHS of (*) $= n^3+11n^2-9n-3 = f(n)$, say.

Now $f(2) > 0$, and $f(n)$ is increasing for $n > 2$.

So, as $n > 1$, $f(n) > 0$, as required, and so $m > \mu$.

(iii) Let M be the mode of X.

$f(x) = kx^n(1-x)$

and $f'(x) = knx^{n-1} - k(n+1)x^n$,

so that $f'(x) = 0$ when $n-(n+1)x = 0$; ie when $x = \frac{n}{n+1}$

Thus $f(x)$ has a local maximum when $x = \frac{n}{n+1}$ (and there are no other stationary points).

Then, as $f(0) = f(1) = 0$, and $f(x) > 0$ for $0 < x < 1$, it follows that the greatest value of $f(x)$ occurs at $x = \frac{n}{n+1}$; ie $M = \frac{n}{n+1}$

From the 1st Part of (ii),

$(n+2)m^{n+1} - (n+1)m^{n+2} = \frac{1}{2}$,

and once again, $M > m$ if $(n+2)M^{n+1} - (n+1)M^{n+2} > \frac{1}{2}$;

ie if $(n+2)(\frac{n}{n+1})^{n+1} - (n+1)(\frac{n}{n+1})^{n+2} > \frac{1}{2}$

$\Leftrightarrow (n+2-n)(\frac{n}{n+1})^{n+1} > \frac{1}{2}$

$\Leftrightarrow (\frac{n}{n+1})^{n+1} > \frac{1}{4}$

$\Leftrightarrow (\frac{n+1}{n})^{n+1} < 4$ (as both sides of the previous inequality are positive)

or $(1 + \frac{1}{n})^{n+1} < 4$ (*)

Now, $(1 + \frac{1}{2})^{2+1} = \frac{27}{8} < 4$,

and given that $(1 + \frac{1}{n})^{n+1}$ is a decreasing function of n,

(*) will hold for $n > 1$, as required; so that $M > m$