STEP 2022, P2, Q1 - Solution (3 pages; 18/7/23)

(i) $\int 2\sqrt{1+x^3}\,dx = 2x\sqrt{1+x^3} - \int 2x.\frac{\frac{1}{2}(3x^2)}{\sqrt{1+x^3}}\,dx$ (by Parts)

so that $\int 2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}}\,dx = 2x\sqrt{1+x^3} + c$

(ii) By Parts: $\int \frac{sinx}{x}\,dx = \frac{1}{x}\,(-cosx) - \int(-cosx).(-1)x^{-2}dx$

$= -\frac{cosx}{x} - \int cosx.x^{-2}\,dx$

$= -\frac{cosx}{x} - [x^{-2}sinx - \int sinx.(-2)x^{-3}\,dx]$

[when integrating by Parts twice, the term that was differentiated in the 1st application of Parts must also be differentiated in the 2nd application, to avoid going round in circles]

$= -\frac{cosx}{x} - \frac{sinx}{x^2} - \int 2\frac{sinx}{x^3}dx$

Hence $\int(x^2+2)\frac{sinx}{x^3}dx = -\frac{cosx}{x} - \frac{sinx}{x^2} + c$

(iii)(a) $y = \frac{e^x}{x}$

<u>Vertical asymptote at $x = 0$</u>

When $x = \delta$ (where δ is a small positive number), $y > 0$;

and when $x = -\delta, y < 0$.

Existence of horizontal asymptote

As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to 0^-$

Stationary points

$$\frac{dy}{dx} = \frac{xe^x - e^x}{x^2} = e^x\frac{(x-1)}{x^2}$$

So there is a stationary point at $x = 1$, when $y = e$.

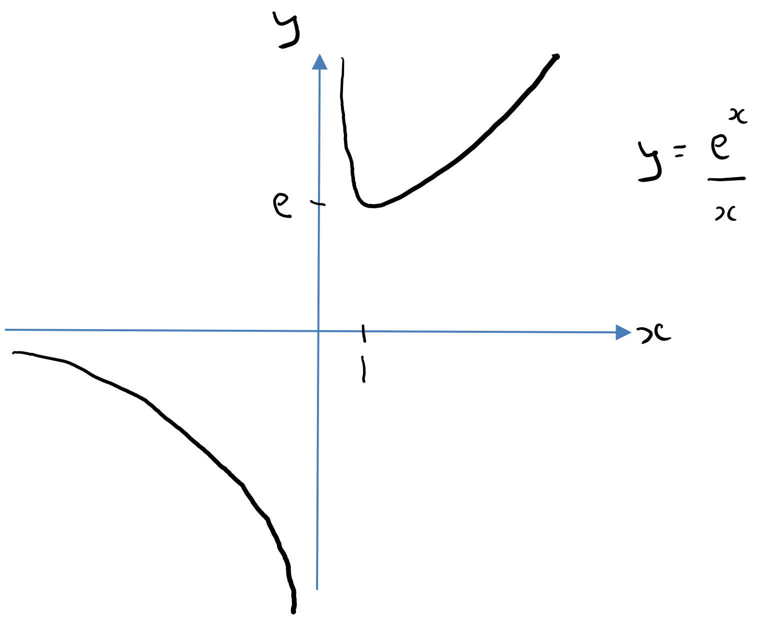
$$\frac{d^2y}{dx^2} = e^x\frac{(x-1)}{x^2} + e^x(-x^{-2} + 2x^{-3}) = e^x\frac{(x^2-2x+2)}{x^3}$$

When $x = 1$, $\frac{d^2y}{dx^2} > 0$, so that there is a minimum at $(1, e)$.

Gradient

Also, $x^2 - 2x + 2 = (x-1)^2 + 1 > 0$, so that $\frac{d^2y}{dx^2} > 0$ (ie

increasing gradient) for $x > 0$, and $\frac{d^2y}{dx^2} < 0$ for $x < 0$



(b) To solve: $\int_a^{2a} \frac{e^x}{x}\, dx = \int_a^{2a} \frac{e^x}{x^2}\, dx$ (*)

Consider $\int_a^{2a} \frac{e^x}{x}\, dx = \left[\frac{1}{x} e^x\right]_a^{2a} - \int_a^{2a} e^x \left(-\frac{1}{x^2}\right) dx$ (by Parts)

Then $\int_a^{2a} \frac{e^x}{x}\, dx = \int_a^{2a} \frac{e^x}{x^2}\, dx \Rightarrow \left[\frac{1}{x} e^x\right]_a^{2a} = 0$

$\Rightarrow \frac{1}{2a} e^{2a} - \frac{1}{a} e^a = 0$

$a = 0$ is one solution of (*)

If $a \neq 0$, $\frac{1}{2} e^a - 1 = 0$, so that $e^a = 2$ and hence $a = ln2$

(c) As in (b), $\int_m^n \frac{e^x}{x}\, dx = \int_m^n \frac{e^x}{x^2}\, dx \Rightarrow \left[\frac{1}{x} e^x\right]_m^n = 0$,

so that $\frac{1}{n} e^n - \frac{1}{m} e^m = 0$, and hence $\frac{1}{n} e^n = \frac{1}{m} e^m$ (**)

From the graph of $y = \frac{e^x}{x}$, distinct m & n satisfying (**) are only possible if $0 < m < 1$, and so there are no distinct integers satisfying (**).