STEP 2022, P2, Q10 - Solution (5 pages; 25/2/24)

(i) Applying the suvat equation $'s = ut + \frac{1}{2}at^{2}'$ separately to horizontal and vertical motion:

At time t from projection, $s = ucos\alpha.t$ and $h = usin\alpha.t - \frac{1}{2}gt^2$

Eliminating t, $h = s.tan\alpha - \frac{1}{2}g(\frac{s}{ucos\alpha})^2$

$= s.tan\alpha - \frac{gs^2}{2u^2cos^2\alpha}$, as required.

(ii) The plane representing the ground can be thought of as the x-y plane, tilted by an angle θ about the x-axis (in the direction of the positive z-axis).

If the cannon is fired in the direction of P, then $s^2 = x^2 + y^2$, and we require the height of P above the x-y plane (ie $ytan\theta$) to be no greater than the maximum h possible. (We note that, for any such point P on the inclined plane, the projectile will be able to reach P, whilst remaining above the plane throughout its motion; ie the only consideration is whether the height of P exceeds the maximum possible.)

Now, $h = stan\alpha - \frac{gs^2}{2u^2}(tan^2\alpha + 1)$

$= -\frac{gs^2}{2u^2}(tan^2\alpha - \frac{2u^2}{gs^2}stan\alpha + 1)$

This is maximised when $tan^2\alpha - \frac{2u^2}{gs^2} stan\alpha + 1$ is minimised;

ie when $(tan\alpha - \frac{u^2}{gs})^2 - \frac{u^4}{g^2s^2} + 1$ is minimised.

This occurs when $tan\alpha - \frac{u^2}{gs} = 0$, and $h = -\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2} + 1)$

So we require $ytan\theta \leq -\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2} + 1)$ (*)

Now, the condition to be proved is

$x^2 + (y + \frac{u^2tan\theta}{g})^2 \leq \frac{u^4sec^2\theta}{g^2}$,

[The fact that the condition we are trying to demonstrate (C, say) is in a different form to (*) can suggest that we haven't derived the condition in the way that was intended by the question setter. It may be worth stopping to see if we have missed a more direct approach. But it might be the case that C is to be used for the next part of the question, and it was intended for (*) to be rearranged to produce C.]

and this is equivalent to

$x^2 + y^2 + \frac{u^4tan^2\theta}{g^2} + 2y.\frac{u^2tan\theta}{g} \leq \frac{u^4}{g^2}(tan^2\theta + 1)$;

or $s^2 + \frac{2yu^2tan\theta}{g} \leq \frac{u^4}{g^2}$, (**)

or $ytan\theta \leq \left(\frac{u^4}{g^2} - s^2\right).\frac{g}{2u^2} = -\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2} + 1)$, which is (*),

as required.

(iii) **1st Part**

[It is easy to overlook the word 'directly' here.]

As the projectile is being fired **directly** up the plane (ie where the gradient of the plane is steepest), $x = 0$.

If the furthest point is $(0, y, ytan\theta)$, then the distance to that point from the cannon is $\sqrt{y^2 + (ytan\theta)^2} = ysec\theta$

As $x = 0$, the condition in (ii) becomes $(y + \frac{u^2 tan\theta}{g})^2 \leq \frac{u^4 sec^2\theta}{g^2}$

or $y + \frac{u^2 tan\theta}{g} \leq \frac{u^2 sec\theta}{g}$ (as both sides of this inequality are positive), and so the maximum value of $ysec\theta$ is

$\frac{u^2}{g}(sec\theta - tan\theta)sec\theta = \frac{u^2(1-sin\theta)}{gcos^2\theta}$

$= \frac{u^2(1-sin\theta)}{g(1-sin^2\theta)} = \frac{u^2}{g(1+sin\theta)}$, as required.

2nd Part

Let the furthest point directly down the plane be $(0, y, ytan\theta)$, where $y = -y'$, with $y' > 0$

The distance from the cannon is $\sqrt{y^2 + (ytan\theta)^2} = y'sec\theta$

and once again $(y + \frac{u^2 tan\theta}{g})^2 \leq \frac{u^4 sec^2\theta}{g^2}$

We want to find the smallest y (ie largest y') that satisfies this inequality.

This occurs when $y+\frac{u^2tan\theta}{g}=-\frac{u^2sec\theta}{g}$,

so that the required distance $y'=-y=\frac{u^2tan\theta}{g}+\frac{u^2sec\theta}{g}$

$=\frac{u^2}{g}(tan\theta+sec\theta)sec\theta=\frac{u^2(sin\theta+1)}{gcos^2\theta}$

$=\frac{u^2(1+sin\theta)}{g(1-sin^2\theta)}=\frac{u^2}{g(1-sin\theta)}$

[Check: This is larger than $\frac{u^2}{g(1+sin\theta)}$, which is to be expected, as gravity is assisting the motion.]

(iv) **1st Part**

With the projectile being fired in the direction of the road, $y=0$, and the distance along the road is x.

The condition in (ii) becomes $x^2+(\frac{u^2tan\theta}{g})^2\leq\frac{u^4sec^2\theta}{g^2}$,

and so $x^2\leq\frac{u^4}{g^2}(sec^2\theta-tan^2\theta)=\frac{u^4}{g^2}$,

and hence the maximum range along the road is $\frac{u^2}{g}$,

making a total length of $\frac{2u^2}{g}$, as the cannon can fire in either direction. [It is easy to overlook this!]

2nd Part

In (ii), we obtained the condition $ytan\theta\leq-\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2}+1)$ (*)

when the cannon was at the Origin, with P being at $(x,y,ytan\theta)$, so that $s^2 = x^2 + y^2$

With the cannon placed instead at the point $(0,\ rcos\theta, rsin\theta)$, this condition becomes $ytan\theta - rsin\theta \leq -\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2} + 1)$,

with $s^2 = x^2 + (y - rcos\theta)^2$ (and P still at $(x, y, ytan\theta)$),

so that $ytan\theta - rsin\theta \leq \frac{u^2}{2g} - \frac{g}{2u^2}[x^2 + (y - rcos\theta)^2]$

In order to maximise the distance along the road, we need to maximise x, with $y = 0$.

So $-rsin\theta \leq \frac{u^2}{2g} - \frac{g}{2u^2}[x^2 + (-rcos\theta)^2]$;

$\frac{g}{2u^2}[x^2 + r^2cos^2\theta\ ] \leq \frac{u^2}{2g} + rsin\theta$;

$x^2 + r^2cos^2\theta \leq \left(\frac{u^2}{2g} + rsin\theta\right)\frac{2u^2}{g}$;

$x^2 \leq \left(\frac{u^2}{2g} + rsin\theta\right)\frac{2u^2}{g} - r^2cos^2\theta$

$= \frac{u^4}{g^2} - (rcos\theta - tan\theta\frac{u^2}{g})^2 + tan^2\theta\frac{u^4}{g^2}$

Thus x is maximised when $rcos\theta - tan\theta\frac{u^2}{g} = 0$,

so that $r = \frac{tan\theta u^2}{gcos\theta}$ or $\frac{tan\theta sec\theta u^2}{g}$,

when $x^2 = \frac{u^4}{g^2}(1 + tan^2\theta) = \frac{u^4}{g^2}sec^2\theta$,

and $x = \frac{u^2sec\theta}{g}$,

making a total length of $\frac{2u^2sec\theta}{g}$, as the cannon can fire in either direction.