STEP 2021, P2, Q7 - Solution (3 pages; 15/2/23)

(i) **1st part**

R can be written $\begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix}$,

and R+I = $\begin{pmatrix} cos\phi + 1 & -sin\phi \\ sin\phi & cos\phi + 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$ for some θ

Then $sin\phi = sin\theta \Rightarrow (a)\ \theta = \phi + 2k\pi$ or $(b)\ \theta = \pi - \phi + 2k\pi$

Then $cos\phi + 1 = cos\theta \Rightarrow$

either, from (a):

$cos\phi + 1 = \cos\,(\phi + 2k\pi)$, so that $cos\phi + 1 = cos\phi$, which has no sol'ns;

or, from (b): $cos\phi + 1 = \cos\,(\pi - \phi + 2k\pi)$,

so that $cos\phi + 1 = \cos(\pi - \phi) = -cos\phi$

and hence $cos\phi = -\frac{1}{2}$, so that $\phi = 120°\ or\ 240°$

2nd part

The effect of R^3 is a rotation of either $3 \times 120° = 360°$; ie I, or

$3 \times 240° = 720°$; ie $2 \times 360°$, and so I also.

(ii) **1st part**

$S^3 = I \Rightarrow \det(S^3) = \det(I) = 1$

As $\det(MN) = detM \times detN$, it follows that $(detS)^3 = 1$, and hence $detS = 1$

2nd part

$$S^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} \quad (*)$$

and $(a+d)S - I = (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} a^2+da-1 & ab+db \\ ac+dc & ad+d^2-1 \end{pmatrix} \quad (1)$$

Then $\det(I) = 1 \Rightarrow ad - bc = 1$,

so that $(1) = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} = S^2$, from (*)

3rd part

$S^2 = (a+d)S - I \Rightarrow S^3 = (a+d)S^2 - S$

$\Rightarrow \mathrm{I} = (\mathrm{a}+\mathrm{d})\{(a+d)S - I\} - S$

$\Rightarrow (\mathrm{a}+\mathrm{d}+1)\mathrm{I} = \{(a+d)^2 - 1\}S = (\mathrm{a}+\mathrm{d}+1)(a+d-1)S$

Then either $a+d+1 = 0$ (**)

or $I = (a+d-1)S$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (a+d-1)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $a+d-1 \neq 0$, and so $b = c = 0$,

and $(a+d-1)a = (a+d-1)d = 1$,

so that $a = d$, and hence $(2a-1)a = 1$

$\Rightarrow 2a^2 - a - 1 = 0$

$\Rightarrow \mathrm{a} = \frac{1 \pm \sqrt{9}}{4} = 1 \; or \; -\frac{1}{2}$

As $S \neq I, a \neq 1$, and so $S = -\frac{1}{2}I$

But then $S^3 = -\frac{1}{8}I \neq I$, which contradicts the fact that $S^3 = I$.

Thus $a + d = -1$, from (**)

(iii) $S \neq I$, as otherwise $S + I = 2I$, but this isn't a rotation; contradicting the fact that $S + I$ is a rotation

Then from (ii), $a + d = -1$,

so that S can be written as $\begin{pmatrix} a & b \\ c & -1-a \end{pmatrix}$

and $S + I$ can be written as $\begin{pmatrix} a+1 & b \\ c & -a \end{pmatrix}$

As this is a rotation (β, say), $cos\beta = a + 1 \;\&\; -a$,

so that $a = -\frac{1}{2}$, and $S + I = \begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix}$,

so that $\beta = 60 \; or \; 300$, giving $sin\beta = \frac{\sqrt{3}}{2} \; or -\frac{\sqrt{3}}{2}$

Then $S = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ or $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, and these are rotations

of 120° and 240°, respectively.

[It is tempting to argue as follows:

$S^3 = I \Rightarrow S^3 - I = 0 \Rightarrow (\mathrm{S} - \mathrm{I})(S^2 + S + I) = 0$

$\Rightarrow$ either S = I or $S^2 + S + I = 0$

But, for matrices it is not true that $AB = 0 \Rightarrow A = 0 \; or \; B = 0$]