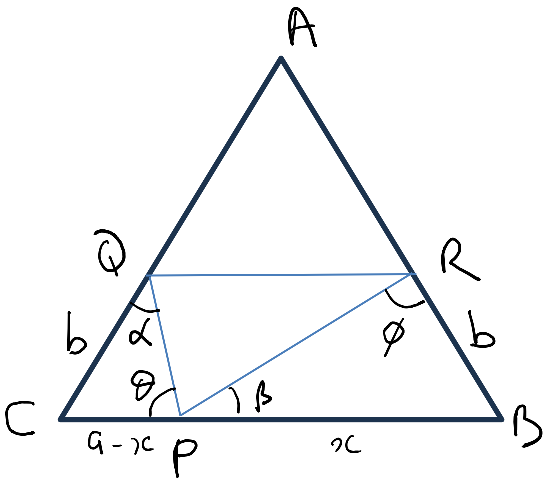
# STEP 2021, P3, Q9 - Solution (4 pages; 6/7/23)

## 1st Part




The result to prove is equivalent to

$\left(\frac{\sqrt{3}}{2}cos\phi + \frac{1}{2}sin\phi\right)\left(\frac{\sqrt{3}}{2}cos\theta + \frac{1}{2}sin\theta\right)x = (a-x)sin\phi sin\theta$

or $\sin\left(\phi + \frac{\pi}{3}\right)\sin\left(\theta + \frac{\pi}{3}\right)x = (a-x)sin\phi sin\theta \quad (*)$

[this suggests use of the Sine Rule]

Referring to the diagram,

$\alpha = \pi - (\theta + \frac{\pi}{3})$ and $\beta = \pi - (\phi + \frac{\pi}{3})$,

so that $sin\alpha = \sin\left(\theta + \frac{\pi}{3}\right)$ and $sin\beta = \sin\left(\phi + \frac{\pi}{3}\right)$

Also, from triangle CPQ, $\frac{sin\alpha}{a-x} = \frac{sin\theta}{b}$,

and from triangle PBR, $\frac{sin\beta}{b} = \frac{sin\phi}{x}$

Then $\frac{sin\alpha}{a-x} = sin\theta.\frac{sin\phi}{xsin\beta}$,

and hence $sin\alpha sin\beta.x = (a-x)sin\phi sin\theta$

and therefore $\sin\left(\phi+\frac{\pi}{3}\right)\sin\left(\theta+\frac{\pi}{3}\right)x=(a-x)sin\phi sin\theta$ , as required.

**2nd Part**

As the frame is smooth, momentum is conserved parallel to CA for the impact at Q, and so $vcos\left(\frac{\pi}{3}\right)=ucos\alpha$, where $u$ is the speed of the ball before impact, and $v$ is its speed afterwards.

Perpendicular to CA, by Newton's law of restitution,

$vsin\left(\frac{\pi}{3}\right)=eusin\alpha$

Hence $\frac{v}{u}=\frac{cos\alpha}{(\frac{1}{2})}=\frac{esin\alpha}{(\frac{\sqrt{3}}{2})}$, so that $e=\sqrt{3}cot\alpha$ (1)

For the impact at R, $wcos\phi=vcos\left(\frac{\pi}{3}\right)$ and $wsin\phi=evsin\left(\frac{\pi}{3}\right)$,

where $w$ is the speed of the ball after impact,

so that $\frac{w}{v}=\frac{(\frac{1}{2})}{cos\phi}=\frac{e(\frac{\sqrt{3}}{2})}{sin\phi}$, so that $e=\frac{tan\phi}{\sqrt{3}}$ (2)

From (2), $\sqrt{3}cot\phi=\frac{1}{e}$, and from (1), $tan\alpha=\frac{\sqrt{3}}{e}$

Also, $tan\alpha=\tan\left(\pi-\left(\theta+\frac{\pi}{3}\right)\right)=-\tan\left(\theta+\frac{\pi}{3}\right)$

$=-\frac{tan\theta+\tan(\frac{\pi}{3})}{1-tan\theta\tan(\frac{\pi}{3})}=\frac{tan\theta+\sqrt{3}}{\sqrt{3}tan\theta-1}$

so that $\frac{\sqrt{3}}{e}\left(\sqrt{3}tan\theta-1\right)=tan\theta+\sqrt{3}$

and hence $tan\theta\left(\frac{3}{e}-1\right)=\sqrt{3}+\frac{\sqrt{3}}{e}$,

so that $\sqrt{3}cot\theta=\frac{\frac{3}{e}-1}{1+\frac{1}{e}}=\frac{3-e}{e+1}$ (*)

Then, substituting for $\sqrt{3}cot\phi$ and $\sqrt{3}cot\theta$ into

$(\sqrt{3}cot\phi+1)(\sqrt{3}cot\theta+1)x=4(a-x)$ gives

$(\frac{1}{e}+1)\left(\frac{3-e}{e+1}+1\right)x=4(a-x)$,

so that $(3-e+e+1)x=4e(a-x)$,

and hence $x[4+4e]=4ea$,

so that $x=\frac{ae}{1+e}$, as required.

**3rd Part**

For the impact at P (assuming the ball continues on to Q again),

$u_1cos\theta=wcos\beta$ and $u_1sin\theta=ewsin\beta$,

where $u_1$ is the speed of the ball after impact,

so that $\frac{u_1}{w}=\frac{cos\beta}{cos\theta}=\frac{esin\beta}{sin\theta}$, so that $e=\frac{tan\theta}{tan\beta}$

and hence $cot\theta=\frac{1}{etan\beta}$ (3)

Now $tan\beta=\tan\left(\pi-\left(\phi+\frac{\pi}{3}\right)\right)=-\tan(\phi+\frac{\pi}{3})$

$=-\frac{tan\phi+\tan(\frac{\pi}{3})}{1-tan\phi\tan(\frac{\pi}{3})}=\frac{tan\phi+\sqrt{3}}{\sqrt{3}tan\phi-1}=\frac{1+\sqrt{3}cot\phi}{\sqrt{3}-cot\phi}=\frac{1+\frac{1}{e}}{\sqrt{3}-\frac{1}{e\sqrt{3}}}$, so that $\sqrt{3}cot\theta = \frac{\sqrt{3}}{e} \cdot \frac{\sqrt{3}-\frac{1}{e\sqrt{3}}}{1+\frac{1}{e}} = \frac{3-\frac{1}{e}}{e+1}$

Comparing this with the expression for $\sqrt{3}cot\theta$ found at (*),

we have that $\frac{3-\frac{1}{e}}{e+1} = \frac{3-e}{e+1}$

So, as $e \neq -1$, $\frac{1}{e} = e$, and hence $e = 1$, as required.