STEP 2021, P3, Q4 - Solution (4 pages; 21/5/23)

(i) [$(\underline{x}.\underline{n})\underline{n}$ and $\underline{x}-(\underline{x}.\underline{n})\underline{n}$ are the perpendicular components of $\underline{x}$, such that one of the components is in the direction of $\underline{n}$, and therefore the other is parallel to the plane.]

Let A be the angle between $\underline{a}$ and $\underline{m}$, and B be the angle between $\underline{b}$ and $\underline{m}$.

Then $\underline{a}.\underline{m} = |\underline{a}||\underline{m}|cosA$ and $\underline{b}.\underline{m} = |\underline{b}||\underline{m}|cosB$,

so that $\frac{cosA}{cosB} = \frac{\underline{a}.\underline{m}}{\underline{b}.\underline{m}}$, as $\underline{a}$ and $\underline{b}$ are of unit length

$=\frac{\underline{a}.\frac{1}{2}(\underline{a}+\underline{b})}{\underline{b}.\frac{1}{2}(\underline{a}+\underline{b})} = \frac{1+\underline{a}.\underline{b}}{1+\underline{b}.\underline{a}} = 1$, so that $cosA = cosB$, and hence $A = B$,

as $0 < \theta < \pi \Rightarrow 0 < A < \pi$ and $0 < B < \pi$

As $\underline{m} = \frac{1}{2}(\underline{a}+\underline{b})$, it lies between $\underline{a}$ and $\underline{b}$, and as $A = B$, $\underline{m}$ therefore bisects the angle between $\underline{a}$ and $\underline{b}$.

(ii) **1st Part**

$\underline{a}_1.\underline{c} = (\underline{a}-(\underline{a}.\underline{c})\underline{c}).\underline{c} = \underline{a}.\underline{c} - (\underline{a}.\underline{c})|\underline{c}|^2 = \underline{a}.\underline{c} - \underline{a}.\underline{c} = 0$, as required (as $|\underline{c}| = 1$)

2nd Part

$|\underline{a}_1|^2 = \underline{a}_1.\underline{a}_1 = (\underline{a}-(\underline{a}.\underline{c})\underline{c}).(\underline{a}-(\underline{a}.\underline{c})\underline{c})$

$= \underline{a}.\underline{a} + (\underline{a}.\underline{c})^2(\underline{c}.\underline{c}) - 2(\underline{a}.\underline{c})^2$

$= 1 + (\underline{a}.\underline{c})^2 - 2(\underline{a}.\underline{c})^2$

$= 1 - (\underline{a}.\underline{c})^2$

$= 1 - (cos\alpha)^2 = sin^2\alpha$

Hence $|\underline{a}_1| = sin\alpha$

3rd Part

$\underline{a}_1.\underline{b}_1 = |\underline{a}_1||\underline{b}_1|cos\phi = sin\alpha sin\beta cos\phi$

(as $|\underline{b}_1| = sin\beta$, by the same method as in the 2nd Part)

Also, $\underline{a}_1.\underline{b}_1 = (\underline{a} - (\underline{a}.\underline{c})\underline{c}).(\underline{b} - (\underline{b}.\underline{c})\underline{c})$

$= \underline{a}.\underline{b} - (\underline{b}.\underline{c})\underline{a}.\underline{c} - (\underline{a}.\underline{c})\underline{c}.\underline{b} + (\underline{a}.\underline{c})(\underline{b}.\underline{c})\underline{c}.\underline{c}$

$= cos\theta - 2cos\beta cos\alpha + cos\alpha cos\beta$

So $sin\alpha sin\beta cos\phi = cos\theta - 2cos\beta cos\alpha + cos\alpha cos\beta$,

and hence $cos\phi = \frac{cos\theta - cos\alpha cos\beta}{sin\alpha sin\beta}$, as required.

(iii) Let C be the angle between $\underline{a}_1$ and $\underline{m}_1$, and D be the angle between $\underline{b}_1$ and $\underline{m}_1$. Then $\underline{m}_1$ bisects $\underline{a}_1$ and $\underline{b}_1$ when $C = D$, provided that $\underline{a}_1$ and $\underline{b}_1$ do not have the same direction; ie provided that $\phi \neq 0$.

So consider separately the two cases:

Case 1: $\phi \neq 0$

Case 2: $\phi = 0$

Now, $cos\theta = \cos(\alpha - \beta) \Leftrightarrow cos\theta - cos\alpha cos\beta = sin\alpha sin\beta$,

so that, from (ii), $cos\phi = \frac{cos\theta - cos\alpha cos\beta}{sin\alpha sin\beta} = 1$, and hence $\phi = 0$

So the result to prove becomes:

$\underline{m}_1$ bisects $\underline{a}_1$ and $\underline{b}_1$ if and only if $\alpha = \beta$ or $\phi = 0$ (***)

Also $\underline{a}_1.\underline{m}_1 = |\underline{a}_1||\underline{m}_1|cosC$ and $\underline{b}_1.\underline{m}_1 = |\underline{b}_1||\underline{m}_1|cosD$ (**)

And $\underline{m}_1 = \underline{m} - (\underline{m}.\underline{c})\underline{c}$

$= \frac{1}{2}(\underline{a} + \underline{b}) - \frac{1}{2}[(\underline{a} + \underline{b}).\underline{c}]\underline{c}$ (*)

And also $\underline{a}_1 = \underline{a} - (\underline{a}.\underline{c})\underline{c}$ and $\underline{b}_1 = \underline{b} - (\underline{b}.\underline{c})\underline{c}$

so that $\underline{a} + \underline{b} = \underline{a}_1 + (\underline{a}.\underline{c})\underline{c} + \underline{b}_1 + (\underline{b}.\underline{c})\underline{c}$

$= \underline{a}_1 + \underline{b}_1 + [(\underline{a} + \underline{b}).\underline{c}]\underline{c}$,

so that $(\underline{a} + \underline{b}) - [(\underline{a} + \underline{b}).\underline{c}]\underline{c} = \underline{a}_1 + \underline{b}_1$,

and hence from (*), $\underline{m}_1 = \frac{1}{2}(\underline{a}_1 + \underline{b}_1)$

For Case 1 ($\phi \neq 0$), from (**):

$$\frac{cosC}{cosD} = \frac{\underline{a}_1.\frac{1}{2}(\underline{a}_1 + \underline{b}_1)sin\beta}{\underline{b}_1.\frac{1}{2}(\underline{a}_1 + \underline{b}_1)sin\alpha} = \frac{(sin^2\alpha + sin\alpha sin\beta cos\phi)sin\beta}{(sin\alpha sin\beta cos\phi + sin^2\beta)sin\alpha}$$

Then $cosC = cosD$, and hence $C = D$ (as both C & D lies between $0°$ and $180°$) when

$(sin^2\alpha + sin\alpha sin\beta cos\phi)sin\beta = (sin\alpha sin\beta cos\phi + sin^2\beta)sin\alpha$;

ie when $sin\alpha + sin\beta cos\phi = sin\alpha cos\phi + sin\beta$;

or $sin\alpha - sin\beta = cos\phi(sin\alpha - sin\beta)$;

ie when $sin\alpha = sin\beta$ or $cos\phi = 1$;

ie when $\alpha = \beta$ (as α & β are acute) or $\phi = 0$

But, as $\phi \neq 0$, we have proved that (for Case 1), $\underline{m}_1$ bisects $\underline{a}_1$ and $\underline{b}_1$ if and only if $\alpha = \beta$, which means that (***) holds.

For Case 2 ($\phi = 0$), $\underline{a}_1 = \underline{b}_1$ and $\underline{m}_1 = \frac{1}{2}(\underline{a}_1 + \underline{b}_1) = \underline{a}_1 = \underline{b}_1$

So $\underline{m}_1$ bisects $\underline{a}_1$ and $\underline{b}_1$ and (***) holds, as $\phi = 0$.